Chapter 4
Differential Relations for Fluid Flow

Motivation. In analyzing fluid motion, we might take one of two paths: (1) seeking an estimate of gross effects (mass flow, induced force, energy change) over a *finite* region or control volume or (2) seeking the point-by-point details of a flow pattern by analyzing an *infinitesimal* region of the flow. The former or gross-average viewpoint was the subject of Chap. 3.

This chapter treats the second in our trio of techniques for analyzing fluid motion: small-scale, or *differential,* analysis. That is, we apply our four basic conservation laws to an infinitesimally small control volume or, alternately, to an infinitesimal fluid system. In either case the results yield the basic *differential equations* of fluid motion. Appropriate *boundary conditions* are also developed.

In their most basic form, these differential equations of motion are quite difficult to solve, and very little is known about their general mathematical properties. However, certain things can be done that have great educational value. First, as shown in Chap. 5, the equations (even if unsolved) reveal the basic dimensionless parameters that govern fluid motion. Second, as shown in Chap. 6, a great number of useful solutions can be found if one makes two simplifying assumptions: (1) steady flow and (2) incompressible flow. A third and rather drastic simplification, frictionless flow, makes our old friend the Bernoulli equation valid and yields a wide variety of idealized, or *perfect-fluid,* possible solutions. These idealized flows are treated in Chap. 8, and we must be careful to ascertain whether such solutions are in fact realistic when compared with actual fluid motion. Finally, even the difficult general differential equations now yield to the approximating technique known as computational fluid dynamics (CFD) whereby the derivatives are simulated by algebraic relations between a finite number of grid points in the flow field, which are then solved on a computer. Reference 1 is an example of a textbook devoted entirely to numerical analysis of fluid motion.

4.1 The Acceleration Field of a Fluid

In Sec. 1.7 we established the cartesian vector form of a velocity field that varies in space and time:

$$\mathbf{V}(\mathbf{r}, t) = \mathbf{i}u(x, y, z, t) + \mathbf{j}v(x, y, z, t) + \mathbf{k}w(x, y, z, t) \tag{1.4}$$

This is the most important variable in fluid mechanics: Knowledge of the velocity vector field is nearly equivalent to *solving* a fluid flow problem. Our coordinates are fixed in space, and we observe the fluid as it passes by—as if we had scribed a set of coordinate lines on a glass window in a wind tunnel. This is the *Eulerian* frame of reference, as opposed to the Lagrangian frame, which follows the moving position of individual particles.

The Eulerian system can be visualized as a window through which we watch a flow. The coordinates (x, y, z) are fixed, and the flow passes by. A fixed instrument placed in the flow takes an Eulerian measurement. In contrast, Lagrangian coordinates follow the moving particles and are common in solid mechanics. Almost all articles and books about fluid mechanics use the Eulerian system. Writers often use *traffic* as an example. A traffic engineer will remain fixed and will measure the flow of cars going by—an Eulerian viewpoint. Conversely, the police will follow specific cars as a function of time—a Lagrangian viewpoint.

To write Newton's second law for an infinitesimal fluid system, we need to calculate the acceleration vector field **a** of the flow. Thus, we compute the total time derivative of the velocity vector:

$$\mathbf{a} = \frac{d\mathbf{V}}{dt} = \mathbf{i}\frac{du}{dt} + \mathbf{j}\frac{dv}{dt} + \mathbf{k}\frac{dw}{dt}$$

Since each scalar component (u, v, w) is a function of the four variables (x, y, z, t), we use the chain rule to obtain each scalar time derivative. For example,

$$\frac{du(x, y, z, t)}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

But, by definition, dx/dt is the local velocity component u, and $dy/dt = v$, and $dz/dt = w$. The total time derivative of u may thus be written as follows, with exactly similar expressions for the time derivatives of v and w:

$$\begin{aligned}
a_x &= \frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \frac{\partial u}{\partial t} + (\mathbf{V}\cdot\nabla)u \\
a_y &= \frac{dv}{dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = \frac{\partial v}{\partial t} + (\mathbf{V}\cdot\nabla)v \\
a_z &= \frac{dw}{dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = \frac{\partial w}{\partial t} + (\mathbf{V}\cdot\nabla)w
\end{aligned} \tag{4.1}$$

Summing these into a vector, we obtain the total acceleration:

$$\mathbf{a} = \frac{d\mathbf{V}}{dt} = \underbrace{\frac{\partial \mathbf{V}}{\partial t}}_{\text{Local}} + \underbrace{\left(u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z}\right)}_{\text{Convective}} = \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V} \tag{4.2}$$

The term $\partial\mathbf{V}/\partial t$ is called the *local acceleration,* which vanishes if the flow is steady—that is, independent of time. The three terms in parentheses are called the *convective* *acceleration,* which arises when the particle moves through regions of spatially varying velocity, as in a nozzle or diffuser. Flows that are nominally "steady" may have large accelerations due to the convective terms.

Note our use of the compact dot product involving **V** and the gradient operator ∇:

$$u\frac{\partial}{\partial x} + \upsilon\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} = \mathbf{V}\cdot\boldsymbol{\nabla} \qquad \text{where} \qquad \boldsymbol{\nabla} = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

The total time derivative—sometimes called the *substantial* or *material* derivative—concept may be applied to any variable, such as the pressure:

$$\frac{dp}{dt} = \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + \upsilon\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z} = \frac{\partial p}{\partial t} + (\mathbf{V}\cdot\boldsymbol{\nabla})p \tag{4.3}$$

Wherever convective effects occur in the basic laws involving mass, momentum, or energy, the basic differential equations become nonlinear and are usually more complicated than flows that do not involve convective changes.

We emphasize that this total time derivative follows a particle of fixed identity, making it convenient for expressing laws of particle mechanics in the eulerian fluid field description. The operator d/dt is sometimes assigned a special symbol such as D/Dt as a further reminder that it contains four terms and follows a fixed particle.

As another reminder of the special nature of d/dt, some writers give it the name *substantial* or *material derivative*.

EXAMPLE 4.1

Given the Eulerian velocity vector field

$$\mathbf{V} = 3t\mathbf{i} + xz\mathbf{j} + ty^2\mathbf{k}$$

find the total acceleration of a particle.

Solution

- *Assumptions:* Given three known unsteady velocity components, $u = 3t$, $\upsilon = xz$, and $w = ty^2$.
- *Approach:* Carry out all the required derivatives with respect to (x, y, z, t), substitute into the total acceleration vector, Eq. (4.2), and collect terms.
- *Solution step 1:* First work out the local acceleration $\partial\mathbf{V}/\partial t$:

$$\frac{\partial \mathbf{V}}{\partial t} = \mathbf{i}\frac{\partial u}{\partial t} + \mathbf{j}\frac{\partial \upsilon}{\partial t} + \mathbf{k}\frac{\partial w}{\partial t} = \mathbf{i}\frac{\partial}{\partial t}(3t) + \mathbf{j}\frac{\partial}{\partial t}(xz) + \mathbf{k}\frac{\partial}{\partial t}(ty^2) = 3\mathbf{i} + 0\mathbf{j} + y^2\,\mathbf{k}$$

- *Solution step 2:* In a similar manner, the convective acceleration terms, from Eq. (4.2), are

$$u\frac{\partial \mathbf{V}}{\partial x} = (3t)\frac{\partial}{\partial x}(3t\mathbf{i} + xz\mathbf{j} + ty^2\mathbf{k}) = (3t)(0\mathbf{i} + z\mathbf{j} + 0\mathbf{k}) = 3tz\,\mathbf{j}$$

$$\upsilon\frac{\partial \mathbf{V}}{\partial y} = (xz)\frac{\partial}{\partial y}(3t\mathbf{i} + xz\mathbf{j} + ty^2\mathbf{k}) = (xz)(0\mathbf{i} + 0\mathbf{j} + 2ty\mathbf{k}) = 2txyz\,\mathbf{k}$$

$$w\frac{\partial \mathbf{V}}{\partial z} = (ty^2)\frac{\partial}{\partial z}(3t\mathbf{i} + xz\mathbf{j} + ty^2\mathbf{k}) = (ty^2)(0\mathbf{i} + x\mathbf{j} + 0\mathbf{k}) = txy^2\,\mathbf{j}$$

- *Solution step 3:* Combine all four terms above into the single "total" or "substantial" derivative:

$$\frac{d\mathbf{V}}{dt} = \frac{\partial \mathbf{V}}{\partial t} + u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z} = (3\mathbf{i} + y^2\mathbf{k}) + 3tz\mathbf{j} + 2txyz\mathbf{k} + txy^2\mathbf{j}$$

$$= 3\mathbf{i} + (3tz + txy^2)\mathbf{j} + (y^2 + 2txyz)\mathbf{k} \qquad \textit{Ans.}$$

- *Comments:* Assuming that **V** is valid everywhere as given, this total acceleration vector $d\mathbf{V}/dt$ applies to all positions and times within the flow field.

4.2 The Differential Equation of Mass Conservation

Conservation of mass, often called the *continuity* relation, states that the fluid mass cannot change. We apply this concept to a very small region. All the basic differential equations can be derived by considering either an elemental control volume or an elemental system. We choose an infinitesimal fixed control volume (dx, dy, dz), as in Fig. 4.1, and use our basic control volume relations from Chap. 3. The flow through each side of the element is approximately one-dimensional, and so the appropriate mass conservation relation to use here is

$$\int_{CV} \frac{\partial \rho}{\partial t}\, d\mathcal{V} + \sum_i (\rho_i A_i V_i)_{\text{out}} - \sum_i (\rho_i A_i V_i)_{\text{in}} = 0 \qquad (3.22)$$

The element is so small that the volume integral simply reduces to a differential term:

$$\int_{CV} \frac{\partial \rho}{\partial t}\, d\mathcal{V} \approx \frac{\partial \rho}{\partial t}\, dx\, dy\, dz$$

The mass flow terms occur on all six faces, three inlets and three outlets. We make use of the field or continuum concept from Chap. 1, where all fluid properties are considered to be uniformly varying functions of time and position, such as $\rho = \rho(x, y, z, t)$. Thus, if T is the temperature on the left face of the element in Fig. 4.1, the right face will have a slightly different temperature $T + (\partial T/\partial x)\, dx$. For mass conservation, if ρu is known on the left face, the value of this product on the right face is $\rho u + (\partial \rho u/\partial x)\, dx$.

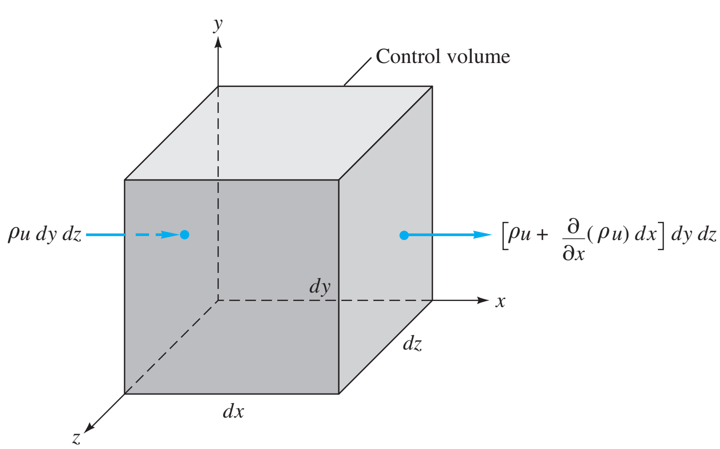


Fig. 4.1 Elemental cartesian fixed control volume showing the inlet and outlet mass flows on the x faces.

Figure 4.1 shows only the mass flows on the x or left and right faces. The flows on the y (bottom and top) and the z (back and front) faces have been omitted to avoid cluttering up the drawing. We can list all these six flows as follows:

Face	Inlet mass flow	Outlet mass flow
x	$\rho u \, dy \, dz$	$\left[\rho u + \frac{\partial}{\partial x}(\rho u)\, dx\right] dy \, dz$
y	$\rho \upsilon \, dx \, dz$	$\left[\rho \upsilon + \frac{\partial}{\partial y}(\rho \upsilon)\, dy\right] dx \, dz$
z	$\rho w \, dx \, dy$	$\left[\rho w + \frac{\partial}{\partial z}(\rho w)\, dz\right] dx \, dy$

Introduce these terms into Eq. (3.22) and we have

$$\frac{\partial \rho}{\partial t} dx\, dy\, dz + \frac{\partial}{\partial x}(\rho u)\, dx\, dy\, dz + \frac{\partial}{\partial y}(\rho \upsilon)\, dx\, dy\, dz + \frac{\partial}{\partial z}(\rho w)\, dx\, dy\, dz = 0$$

The element volume cancels out of all terms, leaving a partial differential equation involving the derivatives of density and velocity:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho \upsilon) + \frac{\partial}{\partial z}(\rho w) = 0 \tag{4.4}$$

This is the desired result: conservation of mass for an infinitesimal control volume. It is often called the *equation of continuity* because it requires no assumptions except that the density and velocity are continuum functions. That is, the flow may be either steady or unsteady, viscous or frictionless, compressible or incompressible.[1] However, the equation does not allow for any source or sink singularities within the element.

The vector gradient operator

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

enables us to rewrite the equation of continuity in a compact form, not that it helps much in finding a solution. The last three terms of Eq. (4.4) are equivalent to the divergence of the vector $\rho\mathbf{V}$

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho \upsilon) + \frac{\partial}{\partial z}(\rho w) \equiv \nabla \cdot (\rho \mathbf{V}) \tag{4.5}$$

so the compact form of the continuity relation is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0 \tag{4.6}$$

In this vector form the equation is still quite general and can readily be converted to other coordinate systems.

[1]One case where Eq. (4.4) might need special care is *two-phase flow,* where the density is discontinuous between the phases. For further details on this case, see Ref. 2, for example.

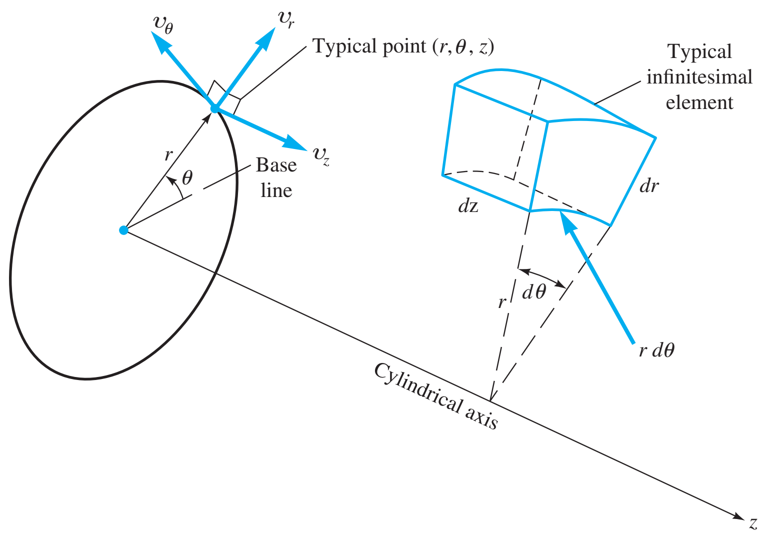


Fig. 4.2 Definition sketch for the cylindrical coordinate system.

Cylindrical Polar Coordinates

The most common alternative to the cartesian system is the *cylindrical polar* coordinate system, sketched in Fig. 4.2. An arbitrary point P is defined by a distance z along the axis, a radial distance r from the axis, and a rotation angle θ about the axis. The three independent orthogonal velocity components are an axial velocity υ_z, a radial velocity υ_r, and a circumferential velocity υ_θ, which is positive counterclockwise—that is, in the direction of increasing θ. In general, all components, as well as pressure and density and other fluid properties, are continuous functions of r, θ, z, and t.

The divergence of any vector function $\mathbf{A}(r, \theta, z, t)$ is found by making the transformation of coordinates

$$r = (x^2 + y^2)^{1/2} \qquad \theta = \tan^{-1}\frac{y}{x} \qquad z = z \tag{4.7}$$

and the result is given here without proof[2]

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(A_\theta) + \frac{\partial}{\partial z}(A_z) \tag{4.8}$$

The general continuity equation (4.6) in cylindrical polar coordinates is thus

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho\upsilon_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho\upsilon_\theta) + \frac{\partial}{\partial z}(\rho\upsilon_z) = 0 \tag{4.9}$$

There are other orthogonal curvilinear coordinate systems, notably *spherical polar* coordinates, which occasionally merit use in a fluid mechanics problem. We shall not treat these systems here except in Prob. P4.12.

There are also other ways to derive the basic continuity equation (4.6) that are interesting and instructive. One example is the use of the divergence theorem. Ask your instructor about these alternative approaches.

[2]See, for example, Ref. 3.

Steady Compressible Flow

If the flow is steady, $\partial/\partial t \equiv 0$ and all properties are functions of position only. Equation (4.6) reduces to

Cartesian: $$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho \upsilon) + \frac{\partial}{\partial z}(\rho w) = 0$$

Cylindrical: $$\frac{1}{r}\frac{\partial}{\partial r}(r\rho \upsilon_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho \upsilon_\theta) + \frac{\partial}{\partial z}(\rho \upsilon_z) = 0 \tag{4.10}$$

Since density and velocity are both variables, these are still nonlinear and rather formidable, but a number of special-case solutions have been found.

Incompressible Flow

A special case that affords great simplification is incompressible flow, where the density changes are negligible. Then $\partial\rho/\partial t \approx 0$ regardless of whether the flow is steady or unsteady, and the density can be slipped out of the divergence in Eq. (4.6) and divided out. The result

$$\nabla \cdot \mathbf{V} = 0 \tag{4.11}$$

is valid for steady or unsteady incompressible flow. The two coordinate forms are

Cartesian: $$\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{4.12a}$$

Cylindrical: $$\frac{1}{r}\frac{\partial}{\partial r}(r\upsilon_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\upsilon_\theta) + \frac{\partial}{\partial z}(\upsilon_z) = 0 \tag{4.12b}$$

These are *linear* differential equations, and a wide variety of solutions are known, as discussed in Chaps. 6 to 8. Since no author or instructor can resist a wide variety of solutions, it follows that a great deal of time is spent studying incompressible flows. Fortunately, this is exactly what should be done, because most practical engineering flows are approximately incompressible, the chief exception being the high-speed gas flows treated in Chap. 9.

When is a given flow approximately incompressible? We can derive a nice criterion by using some density approximations. In essence, we wish to slip the density out of the divergence in Eq. (4.6) and approximate a typical term such as

$$\frac{\partial}{\partial x}(\rho u) \approx \rho \frac{\partial u}{\partial x} \tag{4.13}$$

This is equivalent to the strong inequality

$$\left| u\frac{\partial \rho}{\partial x} \right| \ll \left| \rho \frac{\partial u}{\partial x} \right|$$

or $$\left| \frac{\delta \rho}{\rho} \right| \ll \left| \frac{\delta V}{V} \right| \tag{4.14}$$

As shown in Eq. (1.38), the pressure change is approximately proportional to the density change and the square of the speed of sound a of the fluid:

$$\delta p \approx a^2\, \delta \rho \tag{4.15}$$

Meanwhile, if elevation changes are negligible, the pressure is related to the velocity change by Bernoulli's equation (3.52):

$$\delta p \approx -\rho V \, \delta V \tag{4.16}$$

Combining Eqs. (4.14) to (4.16), we obtain an explicit criterion for incompressible flow:

$$\frac{V^2}{a^2} = \text{Ma}^2 \ll 1 \tag{4.17}$$

where Ma $= V/a$ is the dimensionless *Mach number* of the flow. How small is small? The commonly accepted limit is

$$\text{Ma} \leq 0.3 \tag{4.18}$$

For air at standard conditions, a flow can thus be considered incompressible if the velocity is less than about 100 m/s (330 ft/s). This encompasses a wide variety of airflows: automobile and train motions, light aircraft, landing and takeoff of high-speed aircraft, most pipe flows, and turbomachinery at moderate rotational speeds. Further, it is clear that almost all liquid flows are incompressible, since flow velocities are small and the speed of sound is very large.[3]

Before attempting to analyze the continuity equation, we shall proceed with the derivation of the momentum and energy equations, so that we can analyze them as a group. A very clever device called the *stream function* can often make short work of the continuity equation, but we shall save it until Sec. 4.7.

One further remark is appropriate: The continuity equation is always important and must always be satisfied for a rational analysis of a flow pattern. Any newly discovered momentum or energy "solution" will ultimately fail when subjected to critical analysis if it does not also satisfy the continuity equation.

EXAMPLE 4.2

Under what conditions does the velocity field

$$\mathbf{V} = (a_1x + b_1y + c_1z)\mathbf{i} + (a_2x + b_2y + c_2z)\mathbf{j} + (a_3x + b_3y + c_3z)\mathbf{k}$$

where a_1, b_1, etc. = const, represent an incompressible flow that conserves mass?

Solution

Recalling that $\mathbf{V} = u\mathbf{i} + \upsilon\mathbf{j} + w\mathbf{k}$, we see that $u = (a_1x + b_1y + c_1z)$, etc. Substituting into Eq. (4.12*a*) for incompressible continuity, we obtain

$$\frac{\partial}{\partial x}(a_1x + b_1y + c_1z) + \frac{\partial}{\partial y}(a_2x + b_2y + c_2z) + \frac{\partial}{\partial z}(a_3x + b_3y + c_3z) = 0$$

or $$a_1 + b_2 + c_3 = 0 \qquad \textit{Ans.}$$

At least two of constants a_1, b_2, and c_3 must have opposite signs. Continuity imposes no restrictions whatever on constants b_1, c_1, a_2, c_2, a_3, and b_3, which do not contribute to a volume increase or decrease of a differential element.

[3]An exception occurs in geophysical flows, where a density change is imposed thermally or mechanically rather than by the flow conditions themselves. An example is fresh water layered upon saltwater or warm air layered upon cold air in the atmosphere. We say that the fluid is *stratified,* and we must account for vertical density changes in Eq. (4.6) even if the velocities are small.

EXAMPLE 4.3

An incompressible velocity field is given by

$$u = a(x^2 - y^2) \qquad \upsilon \text{ unknown} \qquad w = b$$

where a and b are constants. What must the form of the velocity component υ be?

Solution

Again Eq. (4.12*a*) applies:

$$\frac{\partial}{\partial x}(ax^2 - ay^2) + \frac{\partial \upsilon}{\partial y} + \frac{\partial b}{\partial z} = 0$$

or

$$\frac{\partial \upsilon}{\partial y} = -2ax \tag{1}$$

This is easily integrated partially with respect to y:

$$\upsilon(x, y, z, t) = -2axy + f(x, z, t) \qquad \textit{Ans.}$$

This is the only possible form for υ that satisfies the incompressible continuity equation. The function of integration f is entirely arbitrary since it vanishes when υ is differentiated with respect to y.[4]

EXAMPLE 4.4

A centrifugal impeller of 40-cm diameter is used to pump hydrogen at 15°C and 1-atm pressure. Estimate the maximum allowable impeller rotational speed to avoid compressibility effects at the blade tips.

Solution

- *Assumptions:* The maximum fluid velocity is approximately equal to the impeller tip speed:

$$V_{max} \approx \Omega r_{max} \qquad \text{where } r_{max} = D/2 = 0.20 \text{ m}$$

- *Approach:* Find the speed of sound of hydrogen and make sure that V_{max} is much less.
- *Property values:* From Table A.4 for hydrogen, $R = 4124 \text{ m}^2/(\text{s}^2 - \text{K})$ and $k = 1.41$. From Eq. (1.39) at 15°C = 288K, compute the speed of sound:

$$a_{H_2} = \sqrt{kRT} = \sqrt{1.41[4124 \text{ m}^2/(\text{s}^2 - \text{K})](288 \text{ K})} \approx 1294 \text{ m/s}$$

- *Final solution step:* Use our rule of thumb, Eq. (4.18), to estimate the maximum impeller speed:

$$V = \Omega r_{max} \le 0.3\text{a} \qquad \text{or} \qquad \Omega(0.2 \text{ m}) \le 0.3(1294 \text{ m/s})$$

$$\text{Solve for} \qquad \Omega \le 1940 \frac{\text{rad}}{\text{s}} \approx 18{,}500 \frac{\text{rev}}{\text{min}} \qquad \textit{Ans.}$$

- *Comments:* This is a high rate because the speed of sound of hydrogen, a light gas, is nearly four times greater than that of air. An impeller moving at this speed in air would create tip shock waves.

[4]This is a very realistic flow that simulates the turning of an inviscid fluid through a 60° angle; see Examples 4.7 and 4.9.

4.3 The Differential Equation of Linear Momentum

This section uses an elemental volume to derive Newton's law for a moving fluid. An alternate approach, which the reader might pursue, would be a force balance on an elemental moving particle. Having done it once in Sec. 4.2 for mass conservation, we can move along a little faster this time. We use the same elemental control volume as in Fig. 4.1, for which the appropriate form of the linear momentum relation is

$$\sum \mathbf{F} = \frac{\partial}{\partial t}\left(\int_{CV} \mathbf{V}\rho \, d\mathcal{V}\right) + \sum (\dot{m}_i \mathbf{V_i})_{\mathbf{out}} - \sum (\dot{m}_i \mathbf{V_i})_{\mathbf{in}} \tag{3.40}$$

Again the element is so small that the volume integral simply reduces to a derivative term:

$$\frac{\partial}{\partial t}(\mathbf{V}\rho \, d\mathcal{V}) \approx \frac{\partial}{\partial t}(\rho \mathbf{V}) \, dx \, dy \, dz \tag{4.19}$$

The momentum fluxes occur on all six faces, three inlets and three outlets. Referring again to Fig. 4.1, we can form a table of momentum fluxes by exact analogy with the discussion that led up to the equation for net mass flux:

Faces	Inlet momentum flux	Outlet momentum flux
x	$\rho u \mathbf{V} \, dy \, dz$	$\left[\rho u \mathbf{V} + \frac{\partial}{\partial x}(\rho u \mathbf{V}) \, dx\right] dy \, dz$
y	$\rho \upsilon \mathbf{V} \, dx \, dz$	$\left[\rho \upsilon \mathbf{V} + \frac{\partial}{\partial y}(\rho \upsilon \mathbf{V}) \, dy\right] dx \, dz$
z	$\rho w \mathbf{V} \, dx \, dy$	$\left[\rho w \mathbf{V} + \frac{\partial}{\partial z}(\rho w \mathbf{V}) \, dz\right] dx \, dy$

Introduce these terms and Eq. (4.19) into Eq. (3.40), and get this intermediate result:

$$\sum \mathbf{F} = dx \, dy \, dz\left[\frac{\partial}{\partial t}(\rho \mathbf{V}) + \frac{\partial}{\partial x}(\rho u \mathbf{V}) + \frac{\partial}{\partial y}(\rho \upsilon \mathbf{V}) + \frac{\partial}{\partial z}(\rho w \mathbf{V})\right] \tag{4.20}$$

Note that this is a vector relation. A simplification occurs if we split up the term in brackets as follows:

$$\begin{aligned} &\frac{\partial}{\partial t}(\rho \mathbf{V}) + \frac{\partial}{\partial x}(\rho u \mathbf{V}) + \frac{\partial}{\partial y}(\rho \upsilon \mathbf{V}) + \frac{\partial}{\partial z}(\rho w \mathbf{V}) \\ &= \mathbf{V}\left[\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{V})\right] + \rho\left(\frac{\partial \mathbf{V}}{\partial t} + u\frac{\partial \mathbf{V}}{\partial x} + \upsilon\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z}\right) \end{aligned} \tag{4.21}$$

The term in brackets on the right-hand side is seen to be the equation of continuity, Eq. (4.6), which vanishes identically. The long term in parentheses on the right-hand side is seen from Eq. (4.2) to be the total acceleration of a particle that instantaneously occupies the control volume:

$$\frac{\partial \mathbf{V}}{\partial t} + u\frac{\partial \mathbf{V}}{\partial x} + \upsilon\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z} = \frac{d\mathbf{V}}{dt} \tag{4.2}$$

Thus, we have now reduced Eq. (4.20) to

$$\sum \mathbf{F} = \rho \frac{d\mathbf{V}}{dt} \, dx \, dy \, dz \tag{4.22}$$

It might be good for you to stop and rest now and think about what we have just done. What is the relation between Eqs. (4.22) and (3.40) for an infinitesimal control volume? Could we have *begun* the analysis at Eq. (4.22)?

Equation (4.22) points out that the net force on the control volume must be of differential size and proportional to the element volume. These forces are of two types, *body* forces and *surface* forces. Body forces are due to external fields (gravity, magnetism, electric potential) that act on the entire mass within the element. The only body force we shall consider in this book is gravity. The gravity force on the differential mass $\rho\, dx\, dy\, dz$ within the control volume is

$$d\mathbf{F}_{\text{grav}} = \rho \mathbf{g}\, dx\, dy\, dz \tag{4.23}$$

where $\mathbf{g}$ may in general have an arbitrary orientation with respect to the coordinate system. In many applications, such as Bernoulli's equation, we take z "up," and $\mathbf{g} = -g\mathbf{k}$.

The surface forces are due to the stresses on the sides of the control surface. These stresses are the sum of hydrostatic pressure plus viscous stresses τ_{ij} that arise from motion with velocity gradients:

$$\sigma_{ij} = \begin{vmatrix} -p + \tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & -p + \tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & -p + \tau_{zz} \end{vmatrix} \tag{4.24}$$

The subscript notation for stresses is given in Fig. 4.3. Unlike velocity $\mathbf{V}$, which is a three-component *vector,* stresses σ_{ij} and τ_{ij} and strain rates ε_{ij} are nine-component *tensors* and require two subscripts to define each component. For further study of *tensor analysis,* see Refs. 6, 11, or 13.

It is not these stresses but their *gradients,* or differences, that cause a net force on the differential control surface. This is seen by referring to Fig. 4.4, which shows only

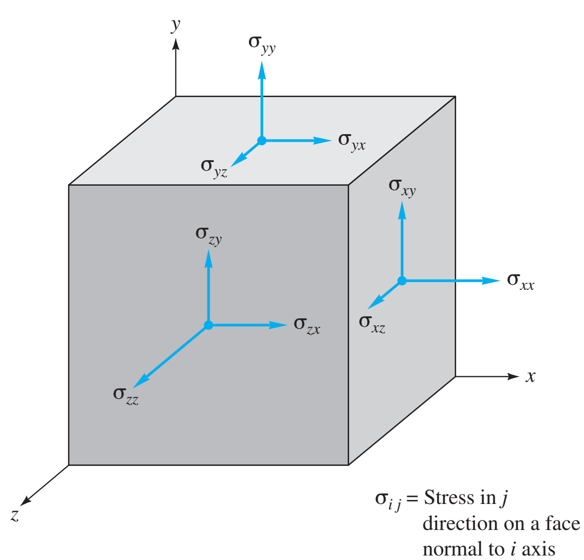


Fig. 4.3 Notation for stresses.

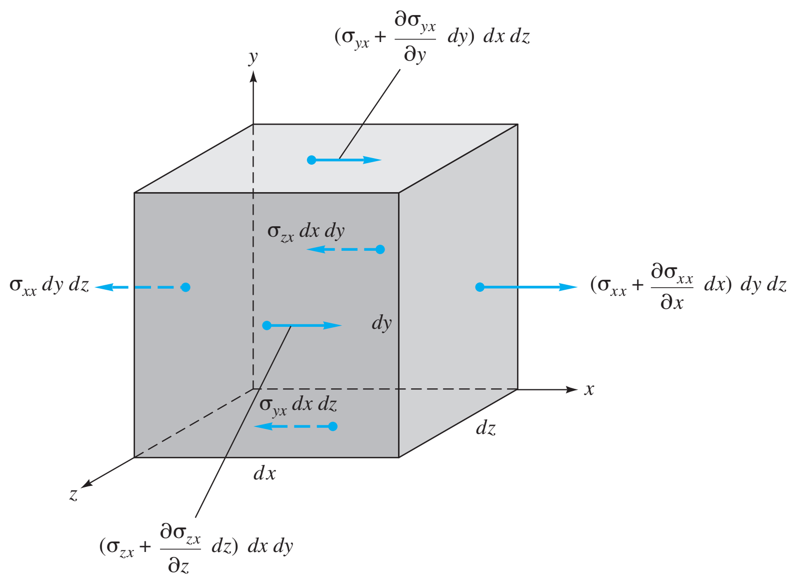


Fig. 4.4 Elemental cartesian fixed control volume showing the surface forces in the x direction only.

the x-directed stresses to avoid cluttering up the drawing. For example, the leftward force $\sigma_{xx}\ dy\ dz$ on the left face is balanced by the rightward force $\sigma_{xx}\ dy\ dz$ on the right face, leaving only the net rightward force $(\partial \sigma_{xx}/\partial x)\ dx\ dy\ dz$ on the right face. The same thing happens on the other four faces, so the net surface force in the x direction is given by

$$dF_{x,\mathrm{surf}} = \left[\frac{\partial}{\partial x}(\sigma_{xx}) + \frac{\partial}{\partial y}(\sigma_{yx}) + \frac{\partial}{\partial z}(\sigma_{zx})\right] dx\ dy\ dz \tag{4.25}$$

We see that this force is proportional to the element volume. Notice that the stress terms are taken from the *top row* of the array in Eq. (4.24). Splitting this row into pressure plus viscous stresses, we can rewrite Eq. (4.25) as

$$\frac{dF_x}{d\mathcal{V}} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}(\tau_{xx}) + \frac{\partial}{\partial y}(\tau_{yx}) + \frac{\partial}{\partial z}(\tau_{zx}) \tag{4.26}$$

where $d\mathcal{V} = dx\ dy\ dz$. In an exactly similar manner, we can derive the y and z forces per unit volume on the control surface:

$$\frac{dF_y}{d\mathcal{V}} = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}(\tau_{xy}) + \frac{\partial}{\partial y}(\tau_{yy}) + \frac{\partial}{\partial z}(\tau_{zy})$$
$$\frac{dF_z}{d\mathcal{V}} = -\frac{\partial p}{\partial z} + \frac{\partial}{\partial x}(\tau_{xz}) + \frac{\partial}{\partial y}(\tau_{yz}) + \frac{\partial}{\partial z}(\tau_{zz}) \tag{4.27}$$

Now we multiply Eqs. (4.26) and (4.27) by **i**, **j**, and **k**, respectively, and add to obtain an expression for the net vector surface force:

$$\left(\frac{d\mathbf{F}}{d\mathcal{V}}\right)_{\mathrm{surf}} = -\nabla p + \left(\frac{d\mathbf{F}}{d\mathcal{V}}\right)_{\mathrm{viscous}} \tag{4.28}$$

where the viscous force has a total of nine terms:

$$\left(\frac{d\mathbf{F}}{d\mathcal{V}}\right)_{\text{viscous}} = \mathbf{i}\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) + \mathbf{j}\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right) + \mathbf{k}\left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right) \tag{4.29}$$

Since each term in parentheses in Eq. (4.29) represents the divergence of a stress component vector acting on the x, y, and z faces, respectively, Eq. (4.29) is sometimes expressed in divergence form:

$$\left(\frac{d\mathbf{F}}{d\mathcal{V}}\right)_{\text{viscous}} = \nabla \cdot \boldsymbol{\tau}_{ij} \tag{4.30}$$

where

$$\boldsymbol{\tau}_{ij} = \begin{bmatrix} \tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \tau_{zz} \end{bmatrix} \tag{4.31}$$

is the viscous stress tensor acting on the element. The surface force is thus the sum of the *pressure gradient* vector and the divergence of the viscous stress tensor. Substituting into Eq. (4.22) and utilizing Eq. (4.23), we have the basic differential momentum equation for an infinitesimal element:

$$\rho \mathbf{g} - \nabla p + \nabla \cdot \boldsymbol{\tau}_{ij} = \rho \frac{d\mathbf{V}}{dt} \tag{4.32}$$

where

$$\frac{d\mathbf{V}}{dt} = \frac{\partial \mathbf{V}}{\partial t} + u\frac{\partial \mathbf{V}}{\partial x} + \upsilon\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z} \tag{4.33}$$

We can also express Eq. (4.32) in words:

Gravity force per unit volume + pressure force per unit volume
+ viscous force per unit volume = density × acceleration (4.34)

Equation (4.32) is so brief and compact that its inherent complexity is almost invisible. It is a *vector* equation, each of whose component equations contains nine terms. Let us therefore write out the component equations in full to illustrate the mathematical difficulties inherent in the momentum equation:

$$\begin{aligned}
\rho g_x - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} &= \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \upsilon\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) \\
\rho g_y - \frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} &= \rho\left(\frac{\partial \upsilon}{\partial t} + u\frac{\partial \upsilon}{\partial x} + \upsilon\frac{\partial \upsilon}{\partial y} + w\frac{\partial \upsilon}{\partial z}\right) \\
\rho g_z - \frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} &= \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + \upsilon\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)
\end{aligned} \tag{4.35}$$

This is the differential momentum equation in its full glory, and it is valid for any fluid in any general motion, particular fluids being characterized by particular viscous stress terms. Note that the last three "convective" terms on the right-hand side of each component equation in (4.35) are nonlinear, which complicates the general mathematical analysis.

Inviscid Flow: Euler's Equation

Equation (4.35) is not ready to use until we write the viscous stresses in terms of velocity components. The simplest assumption is frictionless flow $\tau_{ij} = 0$, for which Eq. (4.32) reduces to

$$\rho \mathbf{g} - \nabla p = \rho \frac{d\mathbf{V}}{dt} \tag{4.36}$$

This is *Euler's equation* for inviscid flow. We show in Sec. 4.9 that Euler's equation can be integrated along a streamline to yield the frictionless Bernoulli equation, (3.52) or (3.54). The complete analysis of inviscid flow fields, using continuity and the Bernoulli relation, is given in Chap. 8.

Newtonian Fluid: Navier-Stokes Equations

For a newtonian fluid, as discussed in Sec. 1.9, the viscous stresses are proportional to the element strain rates and the coefficient of viscosity. For incompressible flow, the generalization of Eq. (1.23) to three-dimensional viscous flow is[5]

$$\tau_{xx} = 2\mu \frac{\partial u}{\partial x} \qquad \tau_{yy} = 2\mu \frac{\partial \upsilon}{\partial y} \qquad \tau_{zz} = 2\mu \frac{\partial w}{\partial z}$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial \upsilon}{\partial x}\right) \qquad \tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \tag{4.37}$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial \upsilon}{\partial z} + \frac{\partial w}{\partial y}\right)$$

where μ is the viscosity coefficient. Substitution into Eq. (4.35) gives the differential momentum equation for a newtonian fluid with constant density and viscosity:

$$\rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \rho \frac{du}{dt}$$

$$\rho g_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 \upsilon}{\partial x^2} + \frac{\partial^2 \upsilon}{\partial y^2} + \frac{\partial^2 \upsilon}{\partial z^2}\right) = \rho \frac{d\upsilon}{dt} \tag{4.38}$$

$$\rho g_z - \frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) = \rho \frac{dw}{dt}$$

These are the incompressible flow *Navier-Stokes equations,* named after C. L. M. H. Navier (1785–1836) and Sir George G. Stokes (1819–1903), who are credited with

[5]When compressibility is significant, additional small terms arise containing the element volume expansion rate and a *second* coefficient of viscosity; see Refs. 4 and 5 for details.

their derivation. They are second-order nonlinear partial differential equations and are quite formidable, but solutions have been found to a variety of interesting viscous flow problems, some of which are discussed in Sec. 4.11 and in Chap. 6 (see also Refs. 4 and 5). For compressible flow, see Eq. (2.29) of Ref. 5.

Equations (4.38) have four unknowns: p, u, υ, and w. They should be combined with the incompressible continuity relation [Eqs. (4.12)] to form four equations in these four unknowns. We shall discuss this again in Sec. 4.6, which presents the appropriate boundary conditions for these equations.

Even though the Navier-Stokes equations have only a limited number of known analytical solutions, they are amenable to fine-gridded computer modeling [1]. The field of CFD is maturing fast, with many commercial software tools available. It is possible now to achieve approximate, but realistic, CFD results for a wide variety of complex two- and three-dimensional viscous flows.

EXAMPLE 4.5

Take the velocity field of Example 4.3, with $b = 0$ for algebraic convenience

$$u = a(x^2 - y^2) \qquad \upsilon = -2axy \qquad w = 0$$

and determine under what conditions it is a solution to the Navier-Stokes momentum equations (4.38). Assuming that these conditions are met, determine the resulting pressure distribution when z is "up" ($g_x = 0$, $g_y = 0$, $g_z = -g$).

Solution

- *Assumptions:* Constant density and viscosity, steady flow (u and υ independent of time).
- *Approach:* Substitute the known (u, υ, w) into Eqs. (4.38) and solve for the pressure gradients. If a unique pressure function $p(x, y, z)$ can then be found, the given solution is exact.
- *Solution step 1:* Substitute (u, υ, w) into Eqs. (4.38) in sequence:

$$\rho(0) - \frac{\partial p}{\partial x} + \mu(2a - 2a + 0) = \rho\left(u\frac{\partial u}{\partial x} + \upsilon\frac{\partial u}{\partial y}\right) = 2a^2\rho\,(x^3 + xy^2)$$

$$\rho(0) - \frac{\partial p}{\partial y} + \mu(0 + 0 + 0) = \rho\left(u\frac{\partial \upsilon}{\partial x} + \upsilon\frac{\partial \upsilon}{\partial y}\right) = 2a^2\rho(x^2y + y^3)$$

$$\rho(-g) - \frac{\partial p}{\partial z} + \mu(0 + 0 + 0) = \rho\left(u\frac{\partial w}{\partial x} + \upsilon\frac{\partial w}{\partial y}\right) = 0$$

Rearrange and solve for the three pressure gradients:

$$\frac{\partial p}{\partial x} = -2a^2\rho(x^3 + xy^2) \qquad \frac{\partial p}{\partial y} = -2a^2\rho(x^2y + y^3) \qquad \frac{\partial p}{\partial z} = -\rho g \tag{1}$$

- *Comment 1:* The vertical pressure gradient is *hydrostatic*. [Could you have predicted this by noting in Eqs. (4.38) that $w = 0$?] However, the pressure is velocity-dependent in the xy plane.

- *Solution step 2:* To determine if the x and y gradients of pressure in Eq. (1) are compatible, evaluate the mixed derivative $(\partial^2 p/\partial x\, \partial y)$; that is, cross-differentiate these two equations:

$$\frac{\partial}{\partial y}\left(\frac{\partial p}{\partial x}\right) = \frac{\partial}{\partial y}[-2a^2\rho(x^3 + xy^2)] = -4a^2\rho xy$$
$$\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial y}\right) = \frac{\partial}{\partial x}[-2a^2\rho(x^2 y + y^3)] = -4a^2\rho xy$$

- *Comment 2:* Since these are equal, the given velocity distribution is indeed an *exact* solution of the Navier-Stokes equations.
- *Solution step 3:* To find the pressure, integrate Eqs. (1), collect, and compare. Start with $\partial p/\partial x$. The procedure requires care! Integrate *partially* with respect to x, holding y and z constant:

$$p = \int \frac{\partial p}{\partial x} dx|_{y,z} = \int -2a^2\rho(x^3 + xy^2)\, dx|_{y,z} = -2a^2\rho\left(\frac{x^4}{4} + \frac{x^2y^2}{2}\right) + f_1(y, z) \quad (2)$$

Note that the "constant" of integration f_1 is a *function* of the variables that were not integrated. Now differentiate Eq. (2) with respect to y and compare with $\partial p/\partial y$ from Eq. (1):

$$\frac{\partial p}{\partial y}|_{(2)} = -2a^2\rho\, x^2 y + \frac{\partial f_1}{\partial y} = \frac{\partial p}{\partial y}|_{(1)} = -2a^2\rho(x^2y + y^3)$$

$$\text{Compare: } \frac{\partial f_1}{\partial y} = -2a^2\rho\, y^3 \quad \text{or} \quad f_1 = \int \frac{\partial f_1}{\partial y} dy|_z = -2a^2\rho\frac{y^4}{4} + f_2(z)$$

$$\text{Collect terms: So far} \quad p = -2a^2\rho\left(\frac{x^4}{4} + \frac{x^2y^2}{2} + \frac{y^4}{4}\right) + f_2(z) \quad (3)$$

This time the "constant" of integration f_2 is a function of z only (the variable not integrated). Now differentiate Eq. (3) with respect to z and compare with $\partial p/\partial z$ from Eq. (1):

$$\frac{\partial p}{\partial z}|_{(3)} = \frac{df_2}{dz} = \frac{\partial p}{\partial z}|_{(1)} = -\rho g \qquad \text{or} \qquad f_2 = -\rho g z + C \quad (4)$$

where C is a constant. This completes our three integrations. Combine Eqs. (3) and (4) to obtain the full expression for the pressure distribution in this flow:

$$p(x, y, z) = -\rho g z - \tfrac{1}{2}a^2\rho(x^4 + y^4 + 2x^2y^2) + C \qquad \textit{Ans. } (5)$$

This is the desired solution. Do you recognize it? Not unless you go back to the beginning and square the velocity components:

$$u^2 + \upsilon^2 + w^2 = V^2 = a^2(x^4 + y^4 + 2x^2y^2) \quad (6)$$

Comparing with Eq. (5), we can rewrite the pressure distribution as

$$p + \tfrac{1}{2}\rho V^2 + pgz = C \quad (7)$$

- *Comment:* This is Bernoulli's equation (3.54). That is no accident, because the velocity distribution given in this problem is one of a family of flows that are solutions to the Navier-Stokes equations and that satisfy Bernoulli's incompressible equation everywhere in the flow field. They are called *irrotational flows,* for which curl $\mathbf{V} = \nabla \times \mathbf{V} \equiv 0$. This subject is discussed again in Sec. 4.9.

4.4 The Differential Equation of Angular Momentum

Having now been through the same approach for both mass and linear momentum, we can go rapidly through a derivation of the differential angular momentum relation. The appropriate form of the integral angular momentum equation for a fixed control volume is

$$\sum \mathbf{M}_o = \frac{\partial}{\partial t}\left[\int_{CV} (\mathbf{r} \times \mathbf{V})\rho \, d\mathcal{V}\right] + \int_{CS} (\mathbf{r} \times \mathbf{V})\rho(\mathbf{V} \cdot \mathbf{n}) \, dA \qquad (3.59)$$

We shall confine ourselves to an axis through O that is parallel to the z axis and passes through the centroid of the elemental control volume. This is shown in Fig. 4.5. Let θ be the angle of rotation about O of the fluid within the control volume. The only stresses that have moments about O are the shear stresses τ_{xy} and τ_{yx}. We can evaluate the moments about O and the angular momentum terms about O. A lot of algebra is involved, and we give here only the result:

$$\left[\tau_{xy} - \tau_{yx} + \frac{1}{2}\frac{\partial}{\partial x}(\tau_{xy})\, dx - \frac{1}{2}\frac{\partial}{\partial y}(\tau_{yx})\, dy\right] dx\, dy\, dz$$
$$= \frac{1}{12}\rho(dx\, dy\, dz)(dx^2 + dy^2)\frac{d^2\theta}{dt^2} \qquad (4.39)$$

Assuming that the angular acceleration $d^2\theta/dt^2$ is not infinite, we can neglect all higher-order differential terms, which leaves a finite and interesting result:

$$\tau_{xy} \approx \tau_{yx} \qquad (4.40)$$

Had we summed moments about axes parallel to y or x, we would have obtained exactly analogous results:

$$\tau_{xz} \approx \tau_{zx} \qquad \tau_{yz} \approx \tau_{zy} \qquad (4.41)$$

There is *no* differential angular momentum equation. Application of the integral theorem to a differential element gives the result, well known to students of stress analysis or strength of materials, that the shear stresses are symmetric: $\tau_{ij} = \tau_{ji}$. This is the

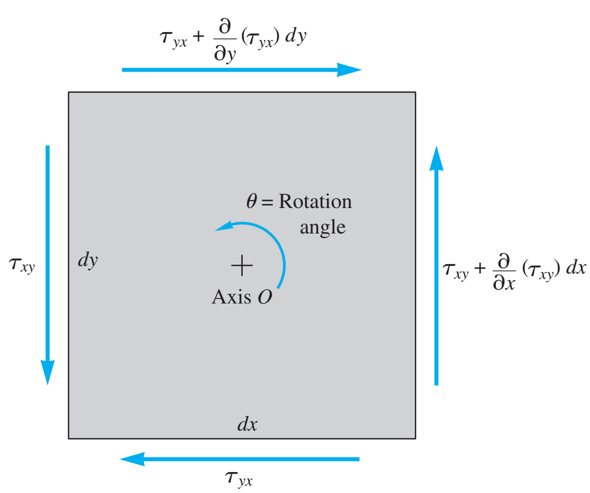


Fig. 4.5 Elemental cartesian fixed control volume showing shear stresses that may cause a net angular acceleration about axis O.

only result of this section.[6] There is no differential equation to remember, which leaves room in your brain for the next topic, the differential energy equation.

4.5 The Differential Equation of Energy[7]

We are now so used to this type of derivation that we can race through the energy equation at a bewildering pace. The appropriate integral relation for the fixed control volume of Fig. 4.1 is

$$\dot{Q} - \dot{W}_s - \dot{W}_v = \frac{\partial}{\partial t}\left(\int_{CV} e\rho \, d\mathcal{V}\right) + \int_{CS}\left(e + \frac{p}{\rho}\right)\rho(\mathbf{V}\cdot\mathbf{n})\, dA \tag{3.66}$$

where $\dot{W}_s = 0$ because there can be no infinitesimal shaft protruding into the control volume. By analogy with Eq. (4.20), the right-hand side becomes, for this tiny element,

$$\dot{Q} - \dot{W}_v = \left[\frac{\partial}{\partial t}(\rho e) + \frac{\partial}{\partial x}(\rho u\zeta) + \frac{\partial}{\partial y}(\rho v\zeta) + \frac{\partial}{\partial z}(\rho w\zeta)\right] dx\, dy\, dz$$

where $\zeta = e + p/\rho$. When we use the continuity equation by analogy with Eq. (4.21), this becomes

$$\dot{Q} - \dot{W}_v = \left(\rho\frac{de}{dt} + \mathbf{V}\cdot\nabla p + p\nabla\cdot\mathbf{V}\right) dx\, dy\, dz \tag{4.42}$$

Thermal Conductivity; Fourier's Law

To evaluate $\dot{Q}$, we neglect radiation and consider only heat conduction through the sides of the element. Experiments for both fluids and solids show that the vector heat transfer per unit area, $\boldsymbol{q}$, is proportional to the vector gradient of temperature, ∇T. This proportionality is called *Fourier's law of conduction,* which is analogous to Newton's viscosity law:

$$\mathbf{q} = -k\nabla T$$

$$\text{or:} \quad q_x = -k\frac{\partial T}{\partial x}, \quad q_y = -k\frac{\partial T}{\partial y}, \quad q_z = -k\frac{\partial T}{\partial z} \tag{4.43}$$

where k is called the *thermal conductivity*, a fluid property that varies with temperature and pressure in much the same way as viscosity. The minus sign satisfies the convention that heat flux is positive in the direction of decreasing temperature. Fourier's law is dimensionally consistent, and k has SI units of joules per (sec-meter-kelvin) and can be correlated with T in much the same way as Eqs. (1.27) and (1.28) for gases and liquids, respectively.

Figure 4.6 shows the heat flow passing through the x faces, the y and z heat flows being omitted for clarity. We can list these six heat flux terms:

[6] We are neglecting the possibility of a finite *couple* being applied to the element by some powerful external force field. See, for example, Ref. 6.

[7] This section may be omitted without loss of continuity.

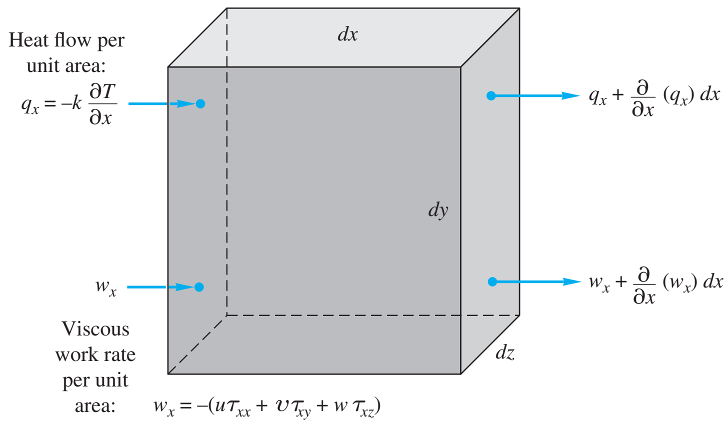


Fig. 4.6 Elemental cartesian control volume showing heat flow and viscous work rate terms in the x direction.

Faces	Inlet heat flux	Outlet heat flux
x	$q_x\,dy\,dz$	$\left[q_x + \frac{\partial}{\partial x}(q_x)\,dx\right]dy\,dz$
y	$q_y\,dx\,dz$	$\left[q_y + \frac{\partial}{\partial y}(q_y)\,dy\right]dx\,dz$
z	$q_z\,dx\,dy$	$\left[q_z + \frac{\partial}{\partial z}(q_z)\,dz\right]dx\,dy$

By adding the inlet terms and subtracting the outlet terms, we obtain the net heat added to the element:

$$\dot{Q} = -\left[\frac{\partial}{\partial x}(q_x) + \frac{\partial}{\partial y}(q_y) + \frac{\partial}{\partial z}(q_z)\right]dx\,dy\,dz = -\nabla\cdot\mathbf{q}\,dx\,dy\,dz \tag{4.44}$$

As expected, the heat flux is proportional to the element volume. Introducing Fourier's law from Eq. (4.43), we have

$$\dot{Q} = \nabla\cdot(k\nabla T)\,dx\,dy\,dz \tag{4.45}$$

The rate of work done by viscous stresses equals the product of the stress component, its corresponding velocity component, and the area of the element face. Figure 4.6 shows the work rate on the left x face is

$$\dot{W}_{\upsilon,\mathrm{LF}} = w_x\,dy\,dz \qquad \text{where } w_x = -(u\tau_{xx} + \upsilon\tau_{xy} + w\tau_{xz}) \tag{4.46}$$

(where the subscript LF stands for left face) and a slightly different work on the right face due to the gradient in w_x. These work fluxes could be tabulated in exactly the same manner as the heat fluxes in the previous table, with w_x replacing q_x, and so on. After outlet terms are subtracted from inlet terms, the net viscous work rate becomes

$$\begin{aligned}\dot{W}_\upsilon = -\bigg[&\frac{\partial}{\partial x}(u\tau_{xx} + \upsilon\tau_{xy} + w\tau_{xz}) + \frac{\partial}{\partial y}(u\tau_{yx} + \upsilon\tau_{yy} + w\tau_{yz})\\ &+ \frac{\partial}{\partial z}(u\tau_{zx} + \upsilon\tau_{zy} + w\tau_{zz})\bigg]dx\,dy\,dz\\ = -&\nabla\cdot(\mathbf{V}\cdot\boldsymbol{\tau}_{ij})\,dx\,dy\,dz\end{aligned} \tag{4.47}$$

We now substitute Eqs. (4.45) and (4.47) into Eq. (4.43) to obtain one form of the differential energy equation:

$$\rho \frac{de}{dt} + \mathbf{V} \cdot \nabla p + p\nabla \cdot \mathbf{V} = \nabla \cdot (k\,\nabla T) + \nabla \cdot (\mathbf{V} \cdot \boldsymbol{\tau}_{ij}) \tag{4.48}$$

where $e = \hat{u} + \frac{1}{2}V^2 + gz$

A more useful form is obtained if we split up the viscous work term:

$$\nabla \cdot (\mathbf{V} \cdot \boldsymbol{\tau}_{ij}) \equiv \mathbf{V} \cdot (\nabla \cdot \boldsymbol{\tau}_{ij}) + \Phi \tag{4.49}$$

where Φ is short for the *viscous-dissipation function*.[8] For a newtonian incompressible viscous fluid, this function has the form

$$\Phi = \mu\left[2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2\right] \tag{4.50}$$

Since all terms are quadratic, viscous dissipation is always positive, so a viscous flow always tends to lose its available energy due to dissipation, in accordance with the second law of thermodynamics.

Now substitute Eq. (4.49) into Eq. (4.48), using the linear momentum equation (4.32) to eliminate $\nabla \cdot \boldsymbol{\tau}_{ij}$. This will cause the kinetic and potential energies to cancel, leaving a more customary form of the general differential energy equation:

$$\rho \frac{d\hat{u}}{dt} + p(\nabla \cdot \mathbf{V}) = \nabla \cdot (k\,\nabla T) + \Phi \tag{4.51}$$

This equation is valid for a newtonian fluid under very general conditions of unsteady, compressible, viscous, heat-conducting flow, except that it neglects radiation heat transfer and internal *sources* of heat that might occur during a chemical or nuclear reaction.

Equation (4.51) is too difficult to analyze except on a digital computer [1]. It is customary to make the following approximations:

$$d\hat{u} \approx c_v\,dT \qquad c_v, \mu, k, \rho \approx \text{const} \tag{4.52}$$

Equation (4.51) then takes the simpler form, for $\nabla \cdot \mathbf{V} = 0$,

$$\rho c_v \frac{dT}{dt} = k\nabla^2 T + \Phi \tag{4.53}$$

which involves temperature T as the sole primary variable plus velocity as a secondary variable through the total time-derivative operator:

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} \tag{4.54}$$

[8]For further details, see, Ref. 5, p. 72.

A great many interesting solutions to Eq. (4.53) are known for various flow conditions, and extended treatments are given in advanced books on viscous flow [4, 5] and books on heat transfer [7, 8].

One well-known special case of Eq. (4.53) occurs when the fluid is at rest or has negligible velocity, where the dissipation Φ and convective terms become negligible:

$$\rho c_p \frac{\partial T}{\partial t} = k\nabla^2 T \tag{4.55}$$

The change from c_v to c_p is correct and justified by the fact that, when pressure terms are neglected from a gas flow energy equation [4, 5], what remains is approximately an enthalpy change, not an internal energy change. This is called the *heat conduction equation* in applied mathematics and is valid for solids and fluids at rest. The solution to Eq. (4.55) for various conditions is a large part of courses and books on heat transfer.

This completes the derivation of the basic differential equations of fluid motion.

4.6 Boundary Conditions for the Basic Equations

There are three basic differential equations of fluid motion, just derived. Let us summarize those here:

Continuity:
$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0 \tag{4.56}$$

Momentum:
$$\rho \frac{d\mathbf{V}}{dt} = \rho \mathbf{g} - \nabla p + \nabla \cdot \boldsymbol{\tau}_{ij} \tag{4.57}$$

Energy:
$$\rho \frac{d\hat{u}}{dt} + p(\nabla \cdot \mathbf{V}) = \nabla \cdot (k\,\nabla T) + \Phi \tag{4.58}$$

where Φ is given by Eq. (4.50). In general, the density is variable, so these three equations contain five unknowns, ρ, V, p, $\hat{u}$, and T. Therefore, we need two additional relations to complete the system of equations. These are provided by data or algebraic expressions for the state relations of the thermodynamic properties:

$$\rho = \rho(p, T) \qquad \hat{u} = \hat{u}(p, T) \tag{4.59}$$

For example, for a perfect gas with constant specific heats, we complete the system with

$$\rho = \frac{p}{RT} \qquad \hat{u} = \int c_v\, dT \approx c_v T + \text{const} \tag{4.60}$$

It is shown in advanced books [4, 5] that this system of equations (4.56) to (4.59) is well posed and can be solved analytically or numerically, subject to the proper boundary conditions.

What are the proper boundary conditions? First, if the flow is unsteady, there must be an *initial condition* or initial spatial distribution known for each variable:

At $t = 0$:
$$\rho, V, p, \hat{u}, T = \text{known}\, f(x, y, z) \tag{4.61}$$

Thereafter, for all times t to be analyzed, we must know something about the variables at each *boundary* enclosing the flow.

Figure 4.7 illustrates the three most common types of boundaries encountered in fluid flow analysis: a solid wall, an inlet or outlet, and a liquid–gas interface.

First, for a solid, impermeable wall, there can be slip and temperature jump in a viscous heat-conducting fluid:

No-slip: $\mathbf{V}_{\text{fluid}} = \mathbf{V}_{\text{wall}} \qquad T_{\text{fluid}} = T_{\text{wall}}$

$$\text{Rarefied gas: } u_{\text{fluid}} - u_{\text{wall}} \approx \ell \frac{\partial u}{\partial n}\Big|_{\text{wall}} \qquad T_{\text{fluid}} - T_{\text{wall}} \approx \left(\frac{2\zeta}{\zeta + 1}\right)\frac{k}{\mu c_p}\ell\frac{\partial T}{\partial n}\Big|_{\text{wall}} \tag{4.62}$$

where, for the rarefied gas, n is normal to the wall, u is parallel to the wall, ℓ is the mean free path of the gas [see Eq. (1.37)], and ζ denotes, just this one time, the specific heat ratio. The above so-called *temperature-jump relation* for gases is given here only for completeness and will not be studied (see page 48 of Ref. 5). A few velocity-jump assignments will be given.

Second, at any inlet or outlet section of the flow, the complete distribution of velocity, pressure, and temperature must be known for all times:

$$\text{Inlet or outlet:} \qquad \text{Known } \mathbf{V}, p, T \tag{4.63}$$

These inlet and outlet sections can be and often are at $\pm\infty$, simulating a body immersed in an infinite expanse of fluid.

Finally, the most complex conditions occur at a liquid–gas interface, or free surface, as sketched in Fig. 4.7. Let us denote the interface by

$$\text{Interface:} \qquad z = \eta(x, y, t) \tag{4.64}$$

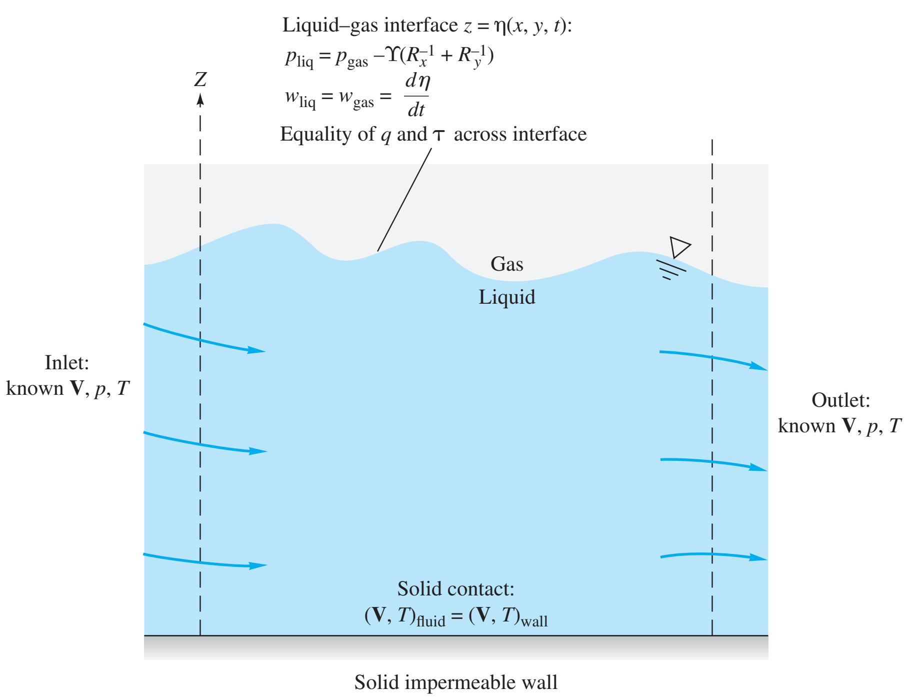


Fig. 4.7 Typical boundary conditions in a viscous heat-conducting fluid flow analysis.

Then there must be equality of vertical velocity across the interface, so that no holes appear between liquid and gas:

$$w_{\text{liq}} = w_{\text{gas}} = \frac{d\eta}{dt} = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + \upsilon\frac{\partial \eta}{\partial y} \tag{4.65}$$

This is called the *kinematic boundary condition*.

There must be mechanical equilibrium across the interface. The viscous shear stresses must balance:

$$(\tau_{zy})_{\text{liq}} = (\tau_{zy})_{\text{gas}} \qquad (\tau_{zx})_{\text{liq}} = (\tau_{zx})_{\text{gas}} \tag{4.66}$$

Neglecting the viscous normal stresses, the pressures must balance at the interface except for surface tension effects:

$$p_{\text{liq}} = p_{\text{gas}} - \Upsilon(R_x^{-1} + R_y^{-1}) \tag{4.67}$$

which is equivalent to Eq. (1.33). The radii of curvature can be written in terms of the free surface position η:

$$R_x^{-1} + R_y^{-1} = \frac{\partial}{\partial x}\left[\frac{\partial\eta/\partial x}{\sqrt{1 + (\partial\eta/\partial x)^2 + (\partial\eta/\partial y)^2}}\right] + \frac{\partial}{\partial y}\left[\frac{\partial\eta/\partial y}{\sqrt{1 + (\partial\eta/\partial x)^2 + (\partial\eta/\partial y)^2}}\right] \tag{4.68}$$

Finally, the heat transfer must be the same on both sides of the interface, since no heat can be stored in the infinitesimally thin interface:

$$(q_z)_{\text{liq}} = (q_z)_{\text{gas}} \tag{4.69}$$

Neglecting radiation, this is equivalent to

$$\left(k\frac{\partial T}{\partial z}\right)_{\text{liq}} = \left(k\frac{\partial T}{\partial z}\right)_{\text{gas}} \tag{4.70}$$

This is as much detail as we wish to give at this level of exposition. Further and even more complicated details on fluid flow boundary conditions are given in Refs. 5 and 9.

Simplified Free Surface Conditions

In the introductory analyses given in this book, such as open-channel flows in Chap. 10, we shall back away from the exact conditions (4.65) to (4.69) and assume that the upper fluid is an "atmosphere" that merely exerts pressure on the lower fluid, with shear and heat conduction negligible. We also neglect nonlinear terms involving the slopes of the free surface. We then have a much simpler and linear set of conditions at the surface:

$$p_{\text{liq}} \approx p_{\text{gas}} - \Upsilon\left(\frac{\partial^2\eta}{\partial x^2} + \frac{\partial^2\eta}{\partial y^2}\right) \qquad w_{\text{liq}} \approx \frac{\partial\eta}{\partial t}$$

$$\left(\frac{\partial V}{\partial z}\right)_{\text{liq}} \approx 0 \qquad \left(\frac{\partial T}{z}\right)_{\text{liq}} \approx 0 \tag{4.71}$$

In many cases, such as open-channel flow, we can also neglect surface tension, so

$$p_{\text{liq}} \approx p_{\text{atm}} \tag{4.72}$$

These are the types of approximations that will be used in Chap. 10. The nondimensional forms of these conditions will also be useful in Chap. 5.

Incompressible Flow with Constant Properties

Flow with constant ρ, μ, and k is a basic simplification that will be used, for example, throughout Chap. 6. The basic equations of motion (4.56) to (4.58) reduce to

Continuity:
$$\nabla \cdot \mathbf{V} = 0 \tag{4.73}$$

Momentum:
$$\rho \frac{d\mathbf{V}}{dt} = \rho \mathbf{g} - \nabla p + \mu \nabla^2 \mathbf{V} \tag{4.74}$$

Energy:
$$\rho c_p \frac{dT}{dt} = k \nabla^2 T + \Phi \tag{4.75}$$

Since ρ is constant, there are only three unknowns: p, $\mathbf{V}$, and T. The system is closed.[9] Not only that, the system splits apart: Continuity and momentum are independent of T. Thus we can solve Eqs. (4.73) and (4.74) entirely separately for the pressure and velocity, using such boundary conditions as

Solid surface:
$$\mathbf{V} = \mathbf{V}_{\text{wall}} \tag{4.76}$$

Inlet or outlet:
$$\text{Known } \mathbf{V}, p \tag{4.77}$$

Free surface:
$$p \approx p_a \qquad w \approx \frac{\partial \eta}{\partial t} \tag{4.78}$$

Later, usually in another course,[10] we can solve for the temperature distribution from Eq. (4.75), which depends on velocity $\mathbf{V}$ through the dissipation Φ and the total time-derivative operator d/dt.

Inviscid Flow Approximations

Chapter 8 assumes inviscid flow throughout, for which the viscosity $\mu = 0$. The momentum equation (4.74) reduces to

$$\rho \frac{d\mathbf{V}}{dt} = \rho \mathbf{g} - \nabla p \tag{4.79}$$

This is *Euler's equation;* it can be integrated along a streamline to obtain Bernoulli's equation (see Sec. 4.9). By neglecting viscosity we have lost the second-order derivative of $\mathbf{V}$ in Eq. (4.74); therefore, we must relax one boundary condition on velocity. The only mathematically sensible condition to drop is the no-slip condition at the wall. We let the flow slip parallel to the wall but do not allow it to flow into the wall. The proper inviscid condition is that the normal velocities must match at any solid surface:

Inviscid flow:
$$(V_n)_{\text{fluid}} = (V_n)_{\text{wall}} \tag{4.80}$$

[9] For this system, what are the thermodynamic equivalents to Eq. (4.59)?

[10] Since temperature is entirely *uncoupled* by this assumption, we may never get around to solving for it here and may ask you to wait until you take a course on heat transfer.

In most cases the wall is fixed; therefore, the proper inviscid flow condition is

$$V_n = 0 \tag{4.81}$$

There is *no* condition whatever on the tangential velocity component at the wall in inviscid flow. The tangential velocity will be part of the solution to an inviscid flow analysis (see Chap. 8).

EXAMPLE 4.6

For steady incompressible laminar flow through a long tube, the velocity distribution is given by

$$\upsilon_z = U\left(1 - \frac{r^2}{R^2}\right) \qquad \upsilon_r = \upsilon_\theta = 0$$

where U is the maximum, or centerline, velocity and R is the tube radius. If the wall temperature is constant at T_w and the temperature $T = T(r)$ only, find $T(r)$ for this flow.

Solution

With $T = T(r)$, Eq. (4.75) reduces for steady flow to

$$\rho c_p \upsilon_r \frac{dT}{dr} = \frac{k}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \mu\left(\frac{d\upsilon_z}{dr}\right)^2 \tag{1}$$

But since $\upsilon_r = 0$ for this flow, the convective term on the left vanishes. Introduce υ_z into Eq. (1) to obtain

$$\frac{k}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = -\mu\left(\frac{d\upsilon_z}{dr}\right)^2 = -\frac{4U^2\mu r^2}{R^4} \tag{2}$$

Multiply through by r/k and integrate once:

$$r\frac{dT}{dr} = -\frac{\mu U^2 r^4}{kR^4} + C_1 \tag{3}$$

Divide through by r and integrate once again:

$$T = -\frac{\mu U^2 r^4}{4kR^4} + C_1 \ln r + C_2 \tag{4}$$

Now we are in position to apply our boundary conditions to evaluate C_1 and C_2.

First, since the logarithm of zero is $-\infty$, the temperature at $r = 0$ will be infinite unless

$$C_1 = 0 \tag{5}$$

Thus, we eliminate the possibility of a logarithmic singularity. The same thing will happen if we apply the *symmetry* condition $dT/dr = 0$ at $r = 0$ to Eq. (3). The constant C_2 is then found by the wall-temperature condition at $r = R$:

$$T = T_w = -\frac{\mu U^2}{4k} + C_2$$

or

$$C_2 = T_w + \frac{\mu U^2}{4k} \tag{6}$$

The correct solution is thus

$$T(r) = T_w + \frac{\mu U^2}{4k}\left(1 - \frac{r^4}{R^4}\right) \qquad \textit{Ans.}\ (7)$$

which is a fourth-order parabolic distribution with a maximum value $T_0 = T_w + \mu U^2/(4k)$ at the centerline.

4.7 The Stream Function

We have seen in Sec. 4.6 that even if the temperature is uncoupled from our system of equations of motion, we must solve the continuity and momentum equations simultaneously for pressure and velocity. The *stream function* ψ is a clever device that allows us to satisfy the continuity equation and then solve the momentum equation directly for the single variable ψ. Lines of constant ψ are streamlines of the flow.

The stream function idea works only if the continuity equation (4.56) can be reduced to *two* terms. In general, we have *four* terms:

Cartesian:
$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho \upsilon) + \frac{\partial}{\partial z}(\rho w) = 0 \tag{4.82a}$$

Cylindrical:
$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho \upsilon_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho \upsilon_\theta) + \frac{\partial}{\partial z}(\rho \upsilon_z) = 0 \tag{4.82b}$$

First, let us eliminate unsteady flow, which is a peculiar and unrealistic application of the stream function idea. Reduce either of Eqs. (4.82) to any *two* terms. The most common application is incompressible flow in the *xy* plane:

$$\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y} = 0 \tag{4.83}$$

This equation is satisfied *identically* if a function $\psi(x, y)$ is defined such that Eq. (4.83) becomes

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) \equiv 0 \tag{4.84}$$

Comparison of (4.83) and (4.84) shows that this new function ψ must be defined such that

$$u = \frac{\partial \psi}{\partial y} \qquad \upsilon = -\frac{\partial \psi}{\partial x} \tag{4.85}$$

or

$$\mathbf{V} = \mathbf{i}\frac{\partial \psi}{\partial y} - \mathbf{j}\frac{\partial \psi}{\partial x}$$

Is this legitimate? Yes, it is just a mathematical trick of replacing two variables (u and υ) by a single higher-order function ψ. The vorticity[11] or curl **V** is an interesting function:

$$\text{curl } \mathbf{V} = -\mathbf{k}\nabla^2\psi \qquad \text{where} \qquad \nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} \tag{4.86}$$

Thus, if we take the curl of the momentum equation (4.74) and utilize Eq. (4.86), we obtain a single equation for ψ for incompressible flow:

$$\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}(\nabla^2\psi) - \frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}(\nabla^2\psi) = \nu\nabla^2(\nabla^2\psi) \tag{4.87}$$

where $\nu = \mu/\rho$ is the kinematic viscosity. This is partly a victory and partly a defeat: Eq. (4.87) is scalar and has only one variable, ψ, but it now contains *fourth*-order derivatives and probably will require computer analysis. There will be four boundary conditions required on ψ. For example, for the flow of a uniform stream in the x direction past a solid body, the four conditions would be

$$\text{At infinity:} \qquad \frac{\partial\psi}{\partial y} = U_\infty \qquad \frac{\partial\psi}{\partial x} = 0$$

$$\text{At the body:} \qquad \frac{\partial\psi}{\partial y} = \frac{\partial\psi}{\partial x} = 0 \tag{4.88}$$

Many examples of numerical solution of Eqs. (4.87) and (4.88) are given in Ref. 1.

One important application is inviscid, incompressible, *irrotational* flow[12] in the xy plane, where curl $\mathbf{V} \equiv 0$. Equations (4.86) and (4.87) reduce to

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0 \tag{4.89}$$

This is the second-order *Laplace equation* (Chap. 8), for which many solutions and analytical techniques are known. Also, boundary conditions like Eq. (4.88) reduce to

$$\text{At infinity:} \qquad \psi = U_\infty y + \text{const} \tag{4.90}$$

$$\text{At the body:} \qquad \psi = \text{const}$$

It is well within our capability to find some useful solutions to Eqs. (4.89) and (4.90), which we shall do in Chap. 8.

Geometric Interpretation of ψ

The fancy mathematics above would serve alone to make the stream function immortal and always useful to engineers. Even better, though, ψ has a beautiful geometric interpretation: Lines of constant ψ are *streamlines* of the flow. This can be shown as follows: From Eq. (1.41) the definition of a streamline in two-dimensional flow is

$$\frac{dx}{u} = \frac{dy}{\upsilon}$$

or

$$u\,dy - \upsilon\,dx = 0 \qquad \text{streamline} \tag{4.91}$$

[11]See Section 4.8.
[12]See Section 4.8.

Introducing the stream function from Eq. (4.85), we have

$$\frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = 0 = d\psi \tag{4.92}$$

Thus the change in ψ is zero along a streamline, or

$$\psi = \text{const along a streamline} \tag{4.93}$$

Having found a given solution $\psi(x, y)$, we can plot lines of constant ψ to give the streamlines of the flow.

There is also a physical interpretation that relates ψ to volume flow. From Fig. 4.8, we can compute the volume flow dQ through an element ds of control surface of unit depth:

$$dQ = (\mathbf{V} \cdot \mathbf{n})\, dA = \left(\mathbf{i}\frac{\partial \psi}{\partial y} - \mathbf{j}\frac{\partial \psi}{\partial x}\right) \cdot \left(\mathbf{i}\frac{dy}{ds} - \mathbf{j}\frac{dx}{ds}\right) ds(1)$$

$$= \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = d\psi \tag{4.94}$$

Thus the change in ψ across the element is numerically equal to the volume flow through the element. The volume flow between any two streamlines in the flow field is equal to the change in stream function between those streamlines:

$$Q_{1\rightarrow 2} = \int_1^2 (\mathbf{V} \cdot \mathbf{n})\, dA = \int_1^2 d\psi = \psi_2 - \psi_1 \tag{4.95}$$

Further, the direction of the flow can be ascertained by noting whether ψ increases or decreases. As sketched in Fig. 4.9, the flow is to the right if ψ_U is greater than ψ_L, where the subscripts stand for upper and lower, as before; otherwise the flow is to the left.

Both the stream function and the velocity potential were invented by the French mathematician Joseph Louis Lagrange and published in his treatise on fluid mechanics in 1781.

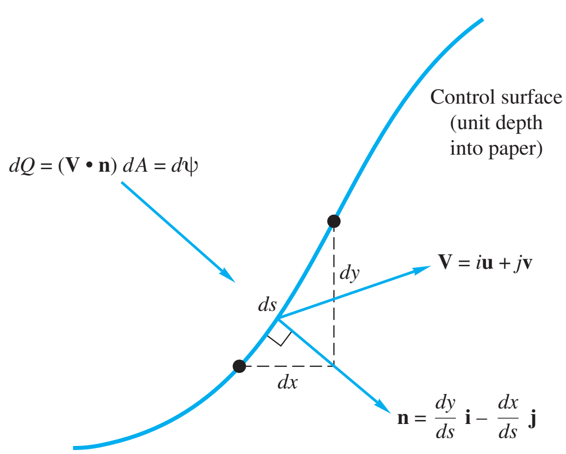


Fig. 4.8 Geometric interpretation of stream function: volume flow through a differential portion of a control surface.

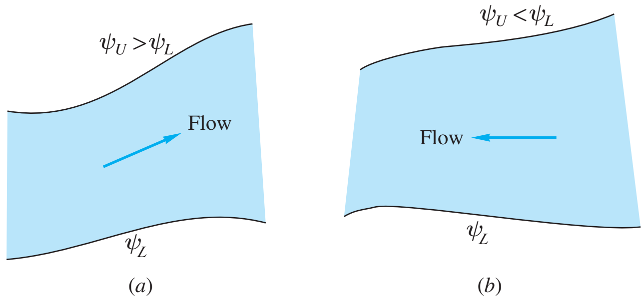


Fig. 4.9 Sign convention for flow in terms of change in stream function: (*a*) flow to the right if ψ_U is greater; (*b*) flow to the left if ψ_L is greater.

EXAMPLE 4.7

If a stream function exists for the velocity field of Example 4.5

$$u = a(x^2 - y^2) \quad \upsilon = -2axy \;\; w = 0$$

find it, plot it, and interpret it.

Solution

- *Assumptions:* Incompressible, two-dimensional flow.
- *Approach:* Use the definition of stream function derivatives, Eqs. (4.85), to find $\psi(x, y)$.
- *Solution step 1:* Note that this velocity distribution was also examined in Example 4.3. It satisfies continuity, Eq. (4.83), but let's check that; otherwise ψ will not exist:

$$\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y} = \frac{\partial}{\partial x}[a(x^2 - y^2)] + \frac{\partial}{\partial y}(-2axy) = 2ax + (-2ax) \equiv 0 \quad \text{checks}$$

Thus we are certain that a stream function exists.

- *Solution step 2:* To find ψ, write out Eqs. (4.85) and integrate:

$$u = \frac{\partial \psi}{\partial y} = ax^2 - ay^2 \tag{1}$$

$$\upsilon = -\frac{\partial \psi}{\partial x} = -2axy \tag{2}$$

and work from either one toward the other. Integrate (1) partially

$$\psi = ax^2 y - \frac{ay^3}{3} + f(x) \tag{3}$$

Differentiate (3) with respect to x and compare with (2)

$$\frac{\partial \psi}{\partial x} = 2axy + f'(x) = 2axy \tag{4}$$

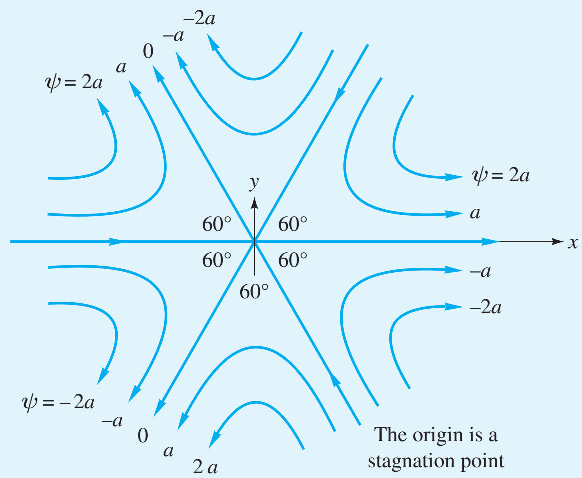
Therefore $f'(x) = 0$, or $f =$ constant. The complete stream function is thus found:

$$\psi = a\left(x^2 y - \frac{y^3}{3}\right) + C \qquad \textit{Ans.} \ (5)$$

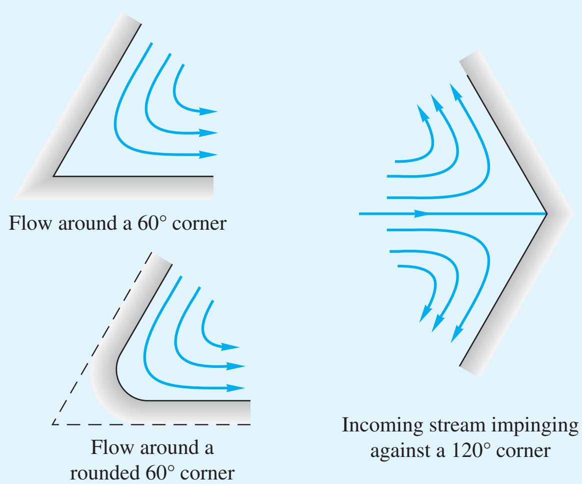
To plot this, set $C = 0$ for convenience and plot the function

$$3x^2y - y^3 = \frac{3\psi}{a} \qquad (6)$$

for constant values of ψ. The result is shown in Fig. E4.7*a* to be six 60° wedges of circulating motion, each with identical flow patterns except for the arrows. Once the streamlines are labeled, the flow directions follow from the sign convention of Fig. 4.9. How can the flow be interpreted? Since there is slip along all streamlines, no streamline can truly represent a solid surface in a viscous flow. However, the flow could represent the impingement of three incoming streams at 60, 180, and 300°. This would be a rather unrealistic yet exact solution to the Navier-Stokes equations, as we showed in Example 4.5.



E4.7*a*



E4.7*b*

By allowing the flow to slip as a frictionless approximation, we could let any given streamline be a body shape. Some examples are shown in Fig. E4.7*b*.

A stream function also exists in a variety of other physical situations where only two coordinates are needed to define the flow. Three examples are illustrated here.

Steady Plane Compressible Flow

Suppose now that the density is variable but that $w = 0$, so that the flow is in the xy plane. Then the equation of continuity becomes

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho \upsilon) = 0 \tag{4.96}$$

We see that this is in exactly the same form as Eq. (4.84). Therefore a compressible flow stream function can be defined such that

$$\rho u = \frac{\partial \psi}{\partial y} \qquad \rho \upsilon = -\frac{\partial \psi}{\partial x} \tag{4.97}$$

Again lines of constant ψ are streamlines of the flow, but the change in ψ is now equal to the *mass* flow, not the volume flow:

$$d\dot{m} = \rho(\mathbf{V} \cdot \mathbf{n})\, dA = d\psi$$

or

$$\dot{m}_{1\to 2} = \int_1^2 \rho(\mathbf{V} \cdot \mathbf{n})\, dA = \psi_2 - \psi_1 \tag{4.98}$$

The sign convention on flow direction is the same as in Fig. 4.9. This particular stream function combines density with velocity and must be substituted into not only momentum but also the energy and state relations (4.58) and (4.59) with pressure and temperature as companion variables. Thus the compressible stream function is not a great victory, and further assumptions must be made to effect an analytical solution to a typical problem (see, for instance, Ref. 5, Chap. 7).

Incompressible Plane Flow in Polar Coordinates

Suppose that the important coordinates are r and θ, with $\upsilon_z = 0$, and that the density is constant. Then Eq. (4.82*b*) reduces to

$$\frac{1}{r}\frac{\partial}{\partial r}(r\upsilon_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\upsilon_\theta) = 0 \tag{4.99}$$

After multiplying through by r, we see that this is the analogous form of Eq. (4.84):

$$\frac{\partial}{\partial r}\left(\frac{\partial \psi}{\partial \theta}\right) + \frac{\partial}{\partial \theta}\left(-\frac{\partial \psi}{\partial r}\right) = 0 \tag{4.100}$$

By comparison of (4.99) and (4.100) we deduce the form of the incompressible polar coordinate stream function:

$$\upsilon_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad \upsilon_\theta = -\frac{\partial \psi}{\partial r} \tag{4.101}$$

Once again lines of constant ψ are streamlines, and the change in ψ is the *volume flow* $Q_{1\to 2} = \psi_2 - \psi_1$. The sign convention is the same as in Fig. 4.9. This type of stream function is very useful in analyzing flows with cylinders, vortices, sources, and sinks (Chap. 8).

Incompressible Axisymmetric Flow

As a final example, suppose that the flow is three-dimensional (υ_r, υ_z) but with no circumferential variations, $\upsilon_\theta = \partial/\partial\theta = 0$ (see Fig. 4.2 for definition of coordinates). Such a flow is termed *axisymmetric,* and the flow pattern is the same when viewed on any meridional plane through the axis of revolution z. For incompressible flow, Eq. (4.82*b*) becomes

$$\frac{1}{r}\frac{\partial}{\partial r}(r\upsilon_r) + \frac{\partial}{\partial z}(\upsilon_z) = 0 \tag{4.102}$$

This doesn't seem to work: Can't we get rid of the one r outside? But when we realize that r and z are independent coordinates, Eq. (4.102) can be rewritten as

$$\frac{\partial}{\partial r}(r\upsilon_r) + \frac{\partial}{\partial z}(r\upsilon_z) = 0 \tag{4.103}$$

By analogy with Eq. (4.84), this has the form

$$\frac{\partial}{\partial r}\left(-\frac{\partial\psi}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial\psi}{\partial r}\right) = 0 \tag{4.104}$$

By comparing (4.103) and (4.104), we deduce the form of an incompressible axisymmetric stream function $\psi(r, z)$

$$\upsilon_r = -\frac{1}{r}\frac{\partial\psi}{\partial z} \qquad \upsilon_z = \frac{1}{r}\frac{\partial\psi}{\partial r} \tag{4.105}$$

Here again lines of constant ψ are streamlines, but there is a factor (2π) in the volume flow: $Q_{1\to2} = 2\pi(\psi_2 - \psi_1)$. The sign convention on flow is the same as in Fig. 4.9.

EXAMPLE 4.8

Investigate the stream function in polar coordinates

$$\psi = U\sin\theta\left(r - \frac{R^2}{r}\right) \tag{1}$$

where U and R are constants, a velocity and a length, respectively. Plot the streamlines. What does the flow represent? Is it a realistic solution to the basic equations?

Solution

The streamlines are lines of constant ψ, which has units of square meters per second. Note that $\psi/(UR)$ is dimensionless. Rewrite Eq. (1) in dimensionless form

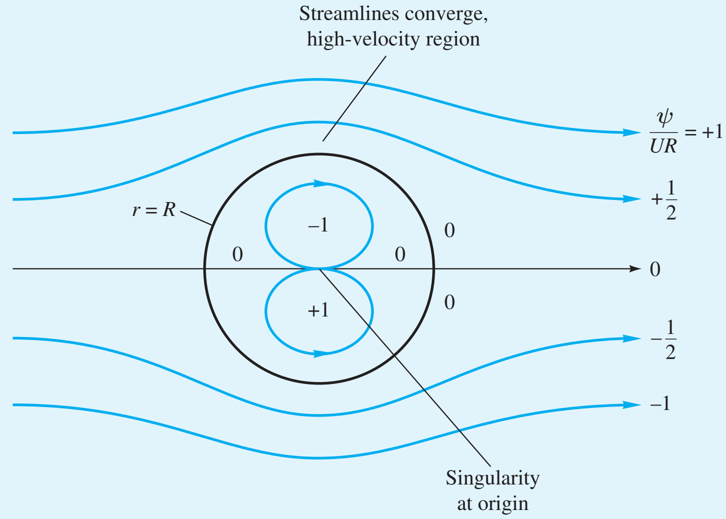
$$\frac{\psi}{UR} = \sin\theta\left(\eta - \frac{1}{\eta}\right) \qquad \eta = \frac{r}{R} \tag{2}$$

Of particular interest is the special line $\psi = 0$. From Eq. (1) or (2) this occurs when (*a*) $\theta = 0$ or 180° and (*b*) $r = R$. Case (*a*) is the x axis, and case (*b*) is a circle of radius R, both of which are plotted in Fig. E4.8.

For any other nonzero value of ψ it is easiest to pick a value of r and solve for θ:

$$\sin\theta = \frac{\psi/(UR)}{r/R - R/r} \qquad (3)$$

In general, there will be two solutions for θ because of the symmetry about the y axis. For example, take $\psi/(UR) = +1.0$:



E4.8

Guess r/R	3.0	2.5	2.0	1.8	1.7	1.618
Compute θ	22° 158°	28° 152°	42° 138°	53° 127°	64° 116°	90°

This line is plotted in Fig. E4.8 and passes over the circle $r = R$. Be careful, though, because there is a second curve for $\psi/(UR) = +1.0$ for small $r < R$ below the x axis:

Guess r/R	0.618	0.6	0.5	0.4	0.3	0.2	0.1
Compute θ	−90°	−70° −110°	−42° −138°	−28° −152°	−19° −161°	−12° −168°	−6° −174°

This second curve plots as a closed curve inside the circle $r = R$. There is a singularity of infinite velocity and indeterminate flow direction at the origin. Figure E4.8 shows the full pattern.

The given stream function, Eq. (1), is an exact and classic solution to the momentum equation (4.38) for frictionless flow. Outside the circle $r = R$ it represents two-dimensional inviscid flow of a uniform stream past a circular cylinder (Sec. 8.4). Inside the circle it represents a rather unrealistic trapped circulating motion of what is called a *line doublet*.

4.8 Vorticity and Irrotationality

The assumption of zero fluid angular velocity, or irrotationality, is a very useful simplification. Here we show that angular velocity is associated with the curl of the local velocity vector.

The differential relations for deformation of a fluid element can be derived by examining Fig. 4.10. Two fluid lines AB and BC, initially perpendicular at time t,

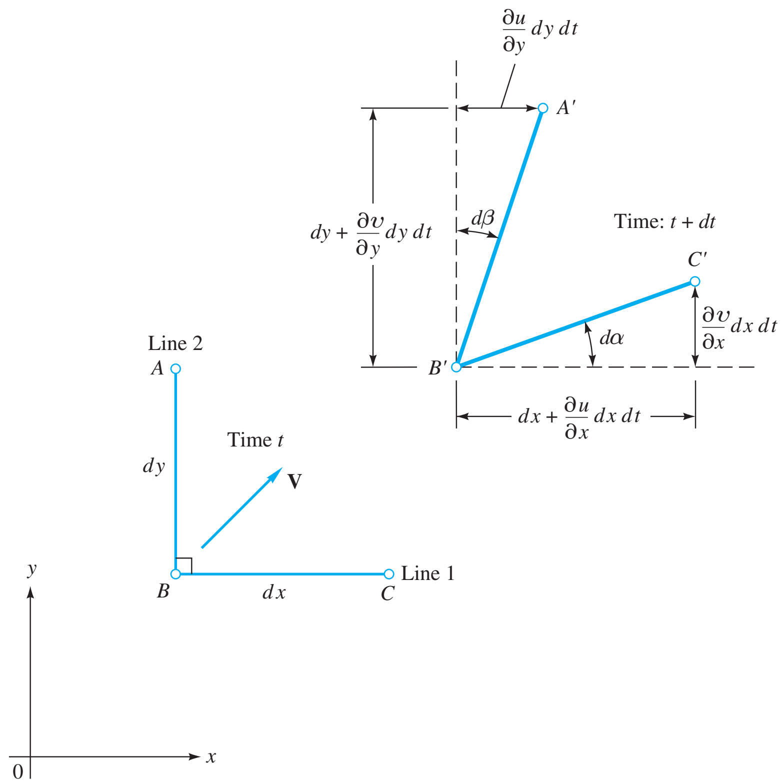


Fig. 4.10 Angular velocity and strain rate of two fluid lines deforming in the xy plane.

move and deform so that at $t + dt$ they have slightly different lengths $A'B'$ and $B'C'$ and are slightly off the perpendicular by angles $d\alpha$ and $d\beta$. Such deformation occurs kinematically because A, B, and C have slightly different velocities when the velocity field **V** has spatial gradients. All these differential changes in the motion of A, B, and C are noted in Fig. 4.10.

We define the angular velocity ω_z about the z axis as the average rate of counterclockwise turning of the two lines:

$$\omega_z = \frac{1}{2}\left(\frac{d\alpha}{dt} - \frac{d\beta}{dt}\right) \tag{4.106}$$

But from Fig. 4.10, $d\alpha$ and $d\beta$ are each directly related to velocity derivatives in the limit of small dt:

$$\begin{aligned} d\alpha &= \lim_{dt\to 0}\left[\tan^{-1}\frac{(\partial \upsilon/\partial x)\,dx\,dt}{dx + (\partial u/\partial x)\,dx\,dt}\right] = \frac{\partial \upsilon}{\partial x}dt \\ d\beta &= \lim_{dt\to 0}\left[\tan^{-1}\frac{(\partial u/\partial y)\,dy\,dt}{dy + (\partial \upsilon/\partial y)\,dy\,dt}\right] = \frac{\partial u}{\partial y}dt \end{aligned} \tag{4.107}$$

Combining Eqs. (4.106) and (4.107) gives the desired result:

$$\omega_z = \frac{1}{2}\left(\frac{\partial \upsilon}{\partial x} - \frac{\partial u}{\partial y}\right) \tag{4.108}$$

In exactly similar manner we determine the other two rates:

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial \upsilon}{\partial z}\right) \qquad \omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \tag{4.109}$$

The vector $\boldsymbol{\omega} = \mathbf{i}\omega_x + \mathbf{j}\omega_y + \mathbf{k}\omega_z$ is thus one-half the curl of the velocity vector

$$\boldsymbol{\omega} = \frac{1}{2}(\text{curl } \mathbf{V}) = \frac{1}{2}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & \upsilon & w \end{vmatrix} \tag{4.110}$$

Since the factor of $\frac{1}{2}$ is annoying, many workers prefer to use a vector twice as large, called the *vorticity*:

$$\boldsymbol{\zeta} = 2\boldsymbol{\omega} = \text{curl } \mathbf{V} \tag{4.111}$$

Many flows have negligible or zero vorticity and are called *irrotational*:

$$\text{curl } \mathbf{V} \equiv 0 \tag{4.112}$$

The next section expands on this idea. Such flows can be incompressible or compressible, steady or unsteady.

We may also note that Fig. 4.10 demonstrates the *shear strain rate* of the element, which is defined as the rate of closure of the initially perpendicular lines:

$$\dot{\varepsilon}_{xy} = \frac{d\alpha}{dt} + \frac{d\beta}{dt} = \frac{\partial \upsilon}{\partial x} + \frac{\partial u}{\partial y} \tag{4.113}$$

When multiplied by viscosity μ, this equals the shear stress τ_{xy} in a newtonian fluid, as discussed earlier in Eqs. (4.37). Appendix D lists strain rate and vorticity components in cylindrical coordinates.

4.9 Frictionless Irrotational Flows

When a flow is both frictionless and irrotational, pleasant things happen. First, the momentum equation (4.38) reduces to Euler's equation:

$$\rho \frac{d\mathbf{V}}{dt} = \rho \mathbf{g} - \boldsymbol{\nabla} p \tag{4.114}$$

Second, there is a great simplification in the acceleration term. Recall from Sec. 4.1 that acceleration has two terms:

$$\frac{d\mathbf{V}}{dt} = \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \boldsymbol{\nabla})\mathbf{V} \tag{4.2}$$

A beautiful vector identity exists for the second term [11]:

$$(\mathbf{V} \cdot \boldsymbol{\nabla})\mathbf{V} \equiv \boldsymbol{\nabla}(\tfrac{1}{2}V^2) + \boldsymbol{\zeta} \times \mathbf{V} \tag{4.115}$$

where ζ = curl $\mathbf{V}$ from Eq. (4.111) is the fluid vorticity.

Now combine (4.114) and (4.115), divide by ρ, and rearrange on the left-hand side. Dot-product the entire equation into an arbitrary vector displacement $d\mathbf{r}$:

$$\left[\frac{\partial \mathbf{V}}{\partial t} + \nabla\left(\frac{1}{2}V^2\right) + \boldsymbol{\zeta} \times \mathbf{V} + \frac{1}{\rho}\nabla p - \mathbf{g}\right] \cdot d\mathbf{r} = 0 \tag{4.116}$$

Nothing works right unless we can get rid of the third term. We want

$$(\boldsymbol{\zeta} \times \mathbf{V}) \cdot (d\mathbf{r}) \equiv 0 \tag{4.117}$$

This will be true under various conditions:

1. $\mathbf{V}$ is zero; trivial, no flow (hydrostatics).
2. $\boldsymbol{\zeta}$ is zero; irrotational flow.
3. $d\mathbf{r}$ is perpendicular to $\boldsymbol{\zeta} \times \mathbf{V}$; this is rather specialized and rare.
4. $d\mathbf{r}$ is parallel to $\mathbf{V}$; we integrate *along a streamline* (see Sec. 3.5).

Condition 4 is the common assumption. If we integrate along a streamline in frictionless compressible flow and take, for convenience, $\mathbf{g} = -g\mathbf{k}$, Eq. (4.116) reduces to

$$\frac{\partial \mathbf{V}}{\partial t} \cdot d\mathbf{r} + d\left(\frac{1}{2}V^2\right) + \frac{dp}{\rho} + g\,dz = 0 \tag{4.118}$$

Except for the first term, these are exact differentials. Integrate between any two points 1 and 2 along the streamline:

$$\int_1^2 \frac{\partial V}{\partial t}\,ds + \int_1^2 \frac{dp}{\rho} + \frac{1}{2}(V_2^2 - V_1^2) + g(z_2 - z_1) = 0 \tag{4.119}$$

where ds is the arc length along the streamline. Equation (4.119) is Bernoulli's equation for frictionless unsteady flow along a streamline and is identical to Eq. (3.53). For incompressible steady flow, it reduces to

$$\frac{p}{\rho} + \frac{1}{2}V^2 + gz = \text{constant along streamline} \tag{4.120}$$

The constant may vary from streamline to streamline unless the flow is also irrotational (assumption 2). For irrotational flow $\zeta = 0$, the offending term Eq. (4.117) vanishes regardless of the direction of $d\mathbf{r}$, and Eq. (4.120) then holds all over the flow field with the same constant.

Velocity Potential

Irrotationality gives rise to a scalar function ϕ similar and complementary to the stream function ψ. From a theorem in vector analysis [11], a vector with zero curl must be the gradient of a scalar function

$$\text{If} \quad \nabla \times \mathbf{V} \equiv 0 \quad \text{then} \quad \mathbf{V} = \nabla\phi \tag{4.121}$$

where $\phi = \phi(x, y, z, t)$ is called the *velocity potential function*. Knowledge of ϕ thus immediately gives the velocity components

$$u = \frac{\partial \phi}{\partial x} \qquad \upsilon = \frac{\partial \phi}{\partial y} \qquad w = \frac{\partial \phi}{\partial z} \tag{4.122}$$

Lines of constant ϕ are called the *potential lines* of the flow.

Note that ϕ, unlike the stream function, is fully three-dimensional and is not limited to two coordinates. It reduces a velocity problem with three unknowns u, υ, and w to a single unknown potential ϕ; many examples are given in Chap. 8. The velocity potential also simplifies the unsteady Bernoulli equation (4.118) because if ϕ exists, we obtain

$$\frac{\partial \mathbf{V}}{\partial t} \cdot d\mathbf{r} = \frac{\partial}{\partial t}(\nabla\phi) \cdot d\mathbf{r} = d\left(\frac{\partial \phi}{\partial t}\right) \tag{4.123}$$

along any arbitrary direction. Equation (4.118) then becomes a relation between ϕ and p:

$$\frac{\partial \phi}{\partial t} + \int \frac{dp}{\rho} + \frac{1}{2}|\nabla\phi|^2 + gz = \text{const} \tag{4.124}$$

This is the unsteady irrotational Bernoulli equation. It is very important in the analysis of accelerating flow fields (see Refs. 10 and 15), but the only application in this text will be in Sec. 9.3 for steady flow.

Orthogonality of Streamlines and Potential Lines

If a flow is both irrotational and described by only two coordinates, ψ and ϕ both exist, and the streamlines and potential lines are everywhere mutually perpendicular except at a stagnation point. For example, for incompressible flow in the xy plane, we would have

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x} \tag{4.125}$$

$$\upsilon = -\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y} \tag{4.126}$$

Can you tell by inspection not only that these relations imply orthogonality but also that ϕ and ψ satisfy Laplace's equation?[13] A line of constant ϕ would be such that the change in ϕ is zero:

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy = 0 = u\,dx + \upsilon\,dy \tag{4.127}$$

Solving, we have

$$\left(\frac{dy}{dx}\right)_{\phi=\text{const}} = -\frac{u}{\upsilon} = -\frac{1}{(dy/dx)_{\psi=\text{const}}} \tag{4.128}$$

Equation (4.128) is the mathematical condition that lines of constant ϕ and ψ be mutually orthogonal. It may not be true at a stagnation point, where both u and υ are zero, so their ratio in Eq. (4.128) is indeterminate.

[13]Equations (4.125) and (4.126) are called the *Cauchy-Riemann equations* and are studied in complex variable theory.

Generation of Rotationality[14]

This is the second time we have discussed Bernoulli's equation under different circumstances (the first was in Sec. 3.5). Such reinforcement is useful, since this is probably the most widely used equation in fluid mechanics. It requires frictionless flow with no shaft work or heat transfer between sections 1 and 2. The flow may or may not be irrotational, the former being an easier condition, allowing a universal Bernoulli constant.

The only remaining question is this: *When* is a flow irrotational? In other words, when does a flow have negligible angular velocity? The exact analysis of fluid rotationality under arbitrary conditions is a topic for advanced study (for example, Ref. 10, Sec. 8.5; Ref. 9, Sec. 5.2; and Ref. 5, Sec. 2.10). We shall simply state those results here without proof.

A fluid flow that is initially irrotational may become rotational if

1. There are significant viscous forces induced by jets, wakes, or solid boundaries. In this case Bernoulli's equation will not be valid in such viscous regions.
2. There are entropy gradients caused by curved shock waves (see Fig. 4.11*b*).
3. There are density gradients caused by *stratification* (uneven heating) rather than by pressure gradients.
4. There are significant *noninertial* effects such as the earth's rotation (the Coriolis acceleration).

In cases 2 to 4, Bernoulli's equation still holds along a streamline if friction is negligible. We shall not study cases 3 and 4 in this book. Case 2 will be treated briefly in Chap. 9 on gas dynamics. Primarily we are concerned with case 1, where rotation is induced by viscous stresses. This occurs near solid surfaces, where the no-slip condition creates a boundary layer through which the stream velocity drops to zero, and in jets and wakes, where streams of different velocities meet in a region of high shear.

Internal flows, such as pipes and ducts, are mostly viscous, and the wall layers grow to meet in the core of the duct. Bernoulli's equation does not hold in such flows unless it is modified for viscous losses.

External flows, such as a body immersed in a stream, are partly viscous and partly inviscid, the two regions being patched together at the edge of the shear layer or boundary layer. Two examples are shown in Fig. 4.11. Figure 4.11*a* shows a low-speed subsonic flow past a body. The approach stream is irrotational; that is, the curl of a constant is zero, but viscous stresses create a rotational shear layer beside and downstream of the body. Generally speaking (see Chap. 7), the shear layer is laminar, or smooth, near the front of the body and turbulent, or disorderly, toward the rear. A separated, or deadwater, region usually occurs near the trailing edge, followed by an unsteady turbulent wake extending far downstream. Some sort of laminar or turbulent viscous theory must be applied to these viscous regions; they are then patched onto the outer flow, which is frictionless and irrotational. If the stream Mach number is less than about 0.3, we can combine Eq. (4.122) with the incompressible continuity equation (4.73):

$$\nabla \cdot \mathbf{V} = \nabla \cdot (\nabla \phi) = 0$$

[14]This section may be omitted without loss of continuity.

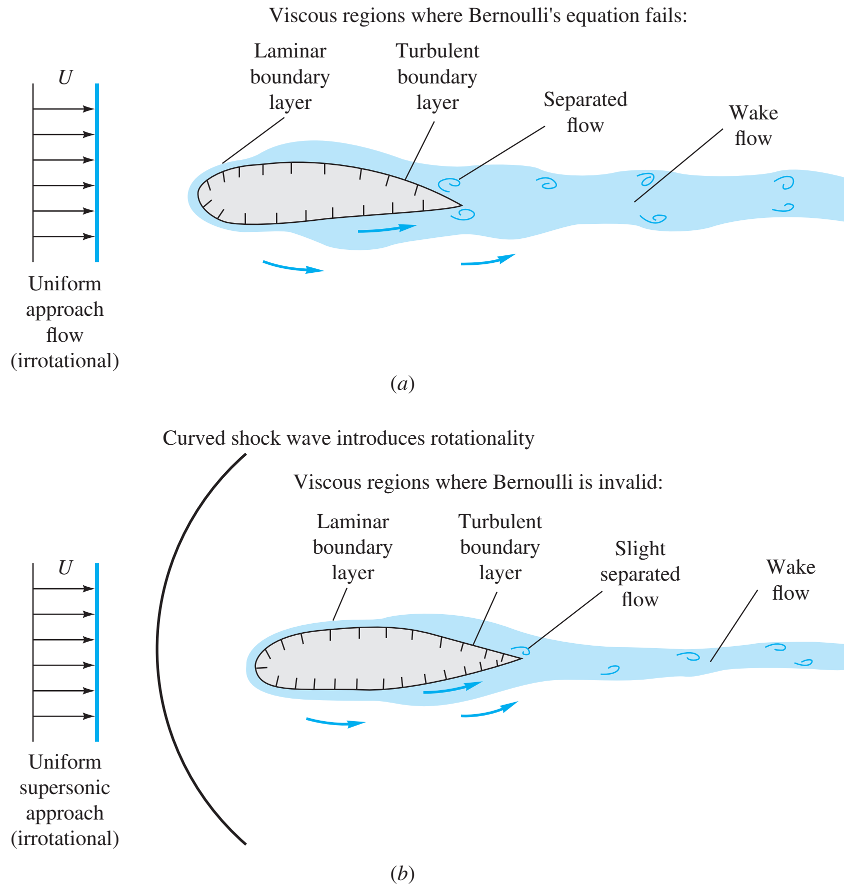


Fig. 4.11 Typical flow patterns illustrating viscous regions patched onto nearly frictionless regions: (*a*) low subsonic flow past a body ($U \ll a$); frictionless, irrotational potential flow outside the boundary layer (Bernoulli and Laplace equations valid); (*b*) supersonic flow past a body ($U > a$); frictionless, rotational flow outside the boundary layer (Bernoulli equation valid, potential flow invalid).

or

$$\nabla^2 \phi = 0 = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \tag{4.129}$$

This is Laplace's equation in three dimensions, there being no restraint on the number of coordinates in potential flow. A great deal of Chap. 8 will be concerned with solving Eq. (4.129) for practical engineering problems; it holds in the entire region of Fig. 4.11*a* outside the shear layer.

Figure 4.11*b* shows a supersonic flow past a round-nosed body. A curved shock wave generally forms in front, and the flow downstream is *rotational* due to entropy gradients (case 2). We can use Euler's equation (4.114) in this frictionless region but not potential theory. The shear layers have the same general character as in Fig. 4.11*a* except that the separation zone is slight or often absent and the wake is usually thinner. Theory of separated flow is presently qualitative, but we can make quantitative estimates of laminar and turbulent boundary layers and wakes.

EXAMPLE 4.9

If a velocity potential exists for the velocity field of Example 4.5

$$u = a(x^2 - y^2) \qquad \upsilon = -2axy \qquad w = 0$$

find it, plot it, and compare with Example 4.7.

Solution

Since $w = 0$, the curl of $\mathbf{V}$ has only one z component, and we must show that it is zero:

$$(\nabla \times \mathbf{V})_z = 2\omega_z = \frac{\partial \upsilon}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial}{\partial x}(-2axy) - \frac{\partial}{\partial y}(ax^2 - ay^2)$$

$$= -2ay + 2ay = 0 \qquad \text{checks} \qquad \textit{Ans.}$$

The flow is indeed irrotational. A velocity potential exists.

To find $\phi(x, y)$, set

$$u = \frac{\partial \phi}{\partial x} = ax^2 - ay^2 \qquad (1)$$

$$\upsilon = \frac{\partial \phi}{\partial y} = -2axy \qquad (2)$$

Integrate (1)

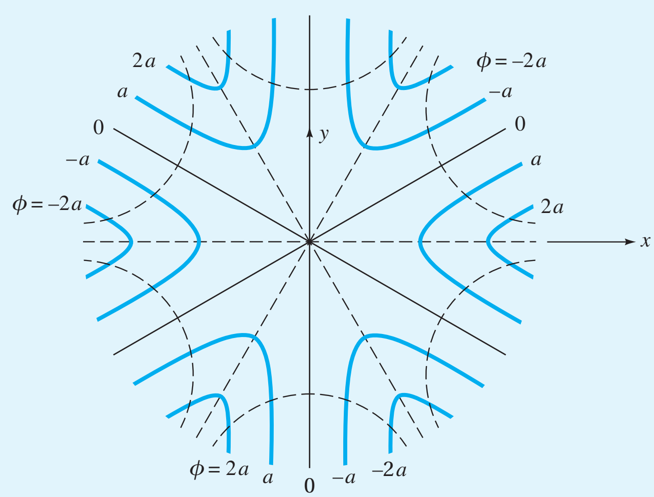
$$\phi = \frac{ax^3}{3} - axy^2 + f(y) \qquad (3)$$

Differentiate (3) and compare with (2)

$$\frac{\partial \phi}{\partial y} = -2axy + f'(y) = -2axy \qquad (4)$$

Therefore $f' = 0$, or $f =$ constant. The velocity potential is

$$\phi = \frac{ax^3}{3} - axy^2 + C \qquad \textit{Ans.}$$



E4.9

Letting $C = 0$, we can plot the ϕ lines in the same fashion as in Example 4.7. The result is shown in Fig. E4.9 (no arrows on ϕ). For this particular problem, the ϕ lines form the same pattern as the ψ lines of Example 4.7 (which are shown here as dashed lines) but are displaced 30°. The ϕ and ψ lines are everywhere perpendicular except at the origin, a stagnation point, where they are 30° apart. We expected trouble at the stagnation point, and there is no general rule for determining the behavior of the lines at that point.

4.10 Some Illustrative Incompressible Viscous Flows

Inviscid flows do *not* satisfy the no-slip condition. They "slip" at the wall but do not flow through the wall. To look at fully viscous no-slip conditions, we must attack the complete Navier-Stokes equation (4.74), and the result is usually not at all irrotational, nor does a velocity potential exist. We look here at three cases: (1) flow between parallel plates due to a moving upper wall, (2) flow between parallel plates due to pressure gradient, and (3) flow between concentric cylinders when the inner one rotates. Other cases will be given as problem assignments or considered in Chap. 6. Extensive solutions for viscous flows are discussed in Refs. 4 and 5. All flows in this section are viscous and rotational.

Couette Flow between a Fixed and a Moving Plate

Consider two-dimensional incompressible plane ($\partial/\partial z = 0$) viscous flow between parallel plates a distance $2h$ apart, as shown in Fig. 4.12. We assume that the plates are very wide and very long, so the flow is essentially axial, $u \neq 0$ but $v = w = 0$. The present case is Fig. 4.12*a*, where the upper plate moves at velocity V but there is no pressure gradient. Neglect gravity effects. We learn from the continuity equation (4.73) that

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 = \frac{\partial u}{\partial x} + 0 + 0 \qquad \text{or} \qquad u = u(y) \text{ only}$$

Thus there is a single nonzero axial velocity component that varies only across the channel. The flow is said to be *fully developed* (far downstream of the entrance).

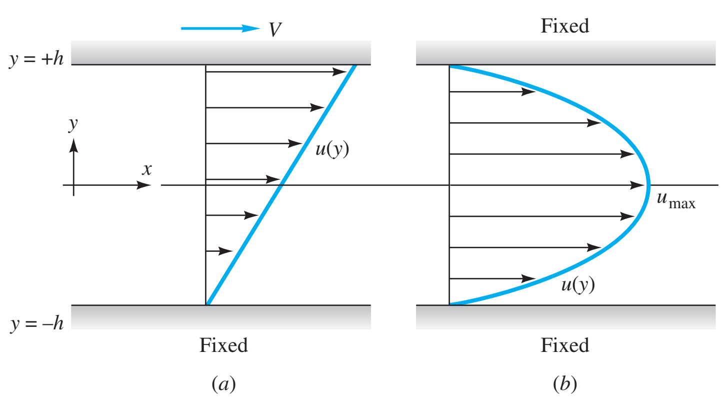


Fig. 4.12 Incompressible viscous flow between parallel plates: (*a*) no pressure gradient, upper plate moving; (*b*) pressure gradient $\partial p/\partial x$ with both plates fixed.

Substitute $u = u(y)$ into the x component of the Navier-Stokes momentum equation (4.74) for two-dimensional (x, y) flow:

$$\rho\left(u\frac{\partial u}{\partial x} + \upsilon\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

or

$$\rho(0 + 0) = 0 + 0 + \mu\left(0 + \frac{d^2u}{dy^2}\right) \tag{4.130}$$

Most of the terms drop out, and the momentum equation reduces to simply

$$\frac{d^2u}{dy^2} = 0 \qquad \text{or} \qquad u = C_1 y + C_2$$

The two constants are found by applying the no-slip condition at the upper and lower plates:

At $y = +h$: $\qquad u = V = C_1 h + C_2$

At $y = -h$: $\qquad u = 0 = C_1(-h) + C_2$

or $\qquad C_1 = \dfrac{V}{2h} \qquad \text{and} \qquad C_2 = \dfrac{V}{2}$

Therefore the solution for this case (a), flow between plates with a moving upper wall, is

$$u = \frac{V}{2h}y + \frac{V}{2} \qquad -h \le y \le +h \tag{4.131}$$

This is *Couette flow* due to a moving wall: a linear velocity profile with no slip at each wall, as anticipated and sketched in Fig. 4.12a. Note that the origin has been placed in the center of the channel for convenience in case (b) which follows.

What we have just presented is a rigorous derivation of the more informally discussed flow of Fig. 1.7 (where y and h were defined differently).

Flow Due to Pressure Gradient between Two Fixed Plates

Case (b) is sketched in Fig. 4.12b. Both plates are fixed ($V = 0$), but the pressure varies in the x direction. If $\upsilon = w = 0$, the continuity equation leads to the same conclusion as case (a)—namely, that $u = u(y)$ only. The x-momentum equation (4.130) changes only because the pressure is variable:

$$\mu\frac{d^2u}{dy^2} = \frac{\partial p}{\partial x} \tag{4.132}$$

Also, since $\upsilon = w = 0$ and gravity is neglected, the y- and z-momentum equations lead to

$$\frac{\partial p}{\partial y} = 0 \quad \text{and} \quad \frac{\partial p}{\partial z} = 0 \quad \text{or} \quad p = p(x) \text{ only}$$

Thus the pressure gradient in Eq. (4.132) is the total and only gradient:

$$\mu\frac{d^2u}{dy^2} = \frac{dp}{dx} = \text{const} < 0 \tag{4.133}$$

Why did we add the fact that dp/dx is *constant*? Recall a useful conclusion from the theory of separation of variables: If two quantities are equal and one varies only with y and the other varies only with x, then they must both equal the same constant. Otherwise they would not be independent of each other.

Why did we state that the constant is *negative*? Physically, the pressure must decrease in the flow direction in order to drive the flow against resisting wall shear stress. Thus the velocity profile $u(y)$ must have negative curvature everywhere, as anticipated and sketched in Fig. 4.12*b*.

The solution to Eq. (4.133) is accomplished by double integration:

$$u = \frac{1}{\mu}\frac{dp}{dx}\frac{y^2}{2} + C_1 y + C_2$$

The constants are found from the no-slip condition at each wall:

$$\text{At } y = \pm h: \qquad u = 0 \qquad \text{or} \qquad C_1 = 0 \qquad \text{and} \qquad C_2 = -\frac{dp}{dx}\frac{h^2}{2\mu}$$

Thus the solution to case (*b*), flow in a channel due to pressure gradient, is

$$u = -\frac{dp}{dx}\frac{h^2}{2\mu}\left(1 - \frac{y^2}{h^2}\right) \tag{4.134}$$

The flow forms a *Poiseuille* parabola of constant negative curvature. The maximum velocity occurs at the centerline $y = 0$:

$$u_{\max} = -\frac{dp}{dx}\frac{h^2}{2\mu} \tag{4.135}$$

Other (laminar) flow parameters are computed in the following example.

EXAMPLE 4.10

For case (*b*) in Fig. 4.12*b*, flow between parallel plates due to the pressure gradient, compute (*a*) the wall shear stress, (*b*) the stream function, (*c*) the vorticity, (*d*) the velocity potential, and (*e*) the average velocity.

Solution

All parameters can be computed from the basic solution, Eq. (4.134), by mathematical manipulation.

Part (a)

The wall shear follows from the definition of a newtonian fluid, Eq. (4.37):

$$\tau_w = \tau_{xy\,\text{wall}} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial \upsilon}{\partial x}\right)\bigg|_{y=\pm h} = \mu\frac{\partial}{\partial y}\left[\left(-\frac{dp}{dx}\right)\left(\frac{h^2}{2\mu}\right)\left(1 - \frac{y^2}{h^2}\right)\right]\bigg|_{y=\pm h}$$

$$= \pm\frac{dp}{dx}h = \mp\frac{2\mu u_{\max}}{h} \qquad \textit{Ans. (a)}$$

The wall shear has the same magnitude at each wall, but by our sign convention of Fig. 4.3, the upper wall has negative shear stress.

Part (b)

Since the flow is plane, steady, and incompressible, a stream function exists:

$$u = \frac{\partial \psi}{\partial y} = u_{\max}\left(1 - \frac{y^2}{h^2}\right) \qquad \upsilon = -\frac{\partial \psi}{\partial x} = 0$$

Integrating and setting $\psi = 0$ at the centerline for convenience, we obtain

$$\psi = u_{\max}\left(y - \frac{y^3}{3h^2}\right) \qquad \textit{Ans. (b)}$$

At the walls, $y = \pm h$ and $\psi = \pm 2u_{\max}h/3$, respectively.

Part (c)

In plane flow, there is only a single nonzero vorticity component:

$$\zeta_z = (\text{curl } \mathbf{V})_z = \frac{\partial \upsilon}{\partial x} - \frac{\partial u}{\partial y} = \frac{2u_{\max}}{h^2} y \qquad \textit{Ans. (c)}$$

The vorticity is highest at the wall and is positive (counterclockwise) in the upper half and negative (clockwise) in the lower half of the fluid. Viscous flows are typically full of vorticity and are not at all irrotational.

Part (d)

From part (*c*), the vorticity is finite. Therefore the flow is not irrotational, and the velocity potential *does not exist*. *Ans.* (*d*)

Part (e)

The average velocity is defined as $V_{av} = Q/A$, where $Q = \int u\, dA$ over the cross section. For our particular distribution $u(y)$ from Eq. (4.134), we obtain

$$V_{av} = \frac{1}{A}\int u\, dA = \frac{1}{b(2h)}\int_{-h}^{+h} u_{\max}\left(1 - \frac{y^2}{h^2}\right) b\, dy = \frac{2}{3}u_{\max} \qquad \textit{Ans. (e)}$$

In plane Poiseuille flow between parallel plates, the average velocity is two-thirds of the maximum (or centerline) value. This result could also have been obtained from the stream function derived in part (*b*). From Eq. (4.95),

$$Q_{\text{channel}} = \psi_{\text{upper}} - \psi_{\text{lower}} = \frac{2u_{\max}h}{3} - \left(-\frac{2u_{\max}h}{3}\right) = \frac{4}{3}u_{\max}h \text{ per unit width}$$

whence $V_{av} = Q/A_{b=1} = (4u_{\max}h/3)/(2h) = 2u_{\max}/3$, the same result.

This example illustrates a statement made earlier: Knowledge of the velocity vector **V** [as in Eq. (4.134)] is essentially the *solution* to a fluid mechanics problem, since all other flow properties can then be calculated.

Fully Developed Laminar Pipe Flow

Perhaps the most useful exact solution of the Navier-Stokes equation is for incompressible flow in a straight circular pipe of radius R, first studied experimentally by G. Hagen in 1839 and J. L. Poiseuille in 1840. By *fully developed* we mean that the region studied is far enough from the entrance that the flow is purely axial, $\upsilon_z \neq 0$, while υ_r and υ_θ are zero. We neglect gravity and also assume axial symmetry—that is, $\partial/\partial\theta = 0$. The equation of continuity in cylindrical coordinates, Eq. (4.12*b*), reduces to

$$\frac{\partial}{\partial z}(\upsilon_z) = 0 \qquad \text{or} \qquad \upsilon_z = \upsilon_z(r) \qquad \text{only}$$

The flow proceeds straight down the pipe without radial motion. The r-momentum equation in cylindrical coordinates, Eq. (D.5), simplifies to $\partial p/\partial r = 0$, or $p = p(z)$ only. The z-momentum equation in cylindrical coordinates, Eq. (D.7), reduces to

$$\rho \upsilon_z \frac{\partial \upsilon_z}{\partial z} = -\frac{dp}{dz} + \mu \nabla^2 \upsilon_z = -\frac{dp}{dz} + \frac{\mu}{r}\frac{d}{dr}\left(r\frac{d\upsilon_z}{dr}\right)$$

The convective acceleration term on the left vanishes because of the previously given continuity equation. Thus the momentum equation may be rearranged as follows:

$$\frac{\mu}{r}\frac{d}{dr}\left(r\frac{d\upsilon_z}{dr}\right) = \frac{dp}{dz} = \text{const} < 0 \tag{4.136}$$

This is exactly the situation that occurred for flow between flat plates in Eq. (4.132). Again the "separation" constant is negative, and pipe flow will look much like the plate flow in Fig. 4.12*b*.

Equation (4.136) is linear and may be integrated twice, with the result

$$\upsilon_z = \frac{dp}{dz}\frac{r^2}{4\mu} + C_1 \ln(r) + C_2$$

where C_1 and C_2 are constants. The boundary conditions are no slip at the wall and finite velocity at the centerline:

$$\text{No slip at } r = R\text{:}\quad \upsilon_z = 0 = \frac{dp}{dz}\frac{R^2}{4\mu} + C_1 \ln(R) + C_2$$

$$\text{Finite velocity at } r = 0\text{:}\quad \upsilon_z = \text{finite} = 0 + C_1 \ln(0) + C_2$$

To avoid a logarithmic singularity, the centerline condition requires that $C_1 = 0$. Then, from no slip, $C_2 = (-dp/dz)(R^2/4\mu)$. The final, and famous, solution for fully developed *Hagen-Poiseuille flow* is

$$\upsilon_z = \left(-\frac{dp}{dz}\right)\frac{1}{4\mu}(R^2 - r^2) \tag{4.137}$$

The velocity profile is a paraboloid with a maximum at the centerline. Just as in Example 4.10, knowledge of the velocity distribution enables other parameters to be calculated:

$$V_{\max} = \upsilon_z(r = 0) = \left(-\frac{dp}{dz}\right)\frac{R^2}{4\mu}$$

$$V_{\text{avg}} = \frac{1}{A}\int \upsilon_z dA = \frac{1}{\pi R^2}\int_0^R V_{\max}\left(1 - \frac{r^2}{R^2}\right) 2\pi r\, dr = \frac{V_{\max}}{2} = \left(-\frac{dp}{dz}\right)\frac{R^2}{8\mu}$$

$$Q = \int \upsilon_z dA = \int_0^R V_{\max}\left(1 - \frac{r^2}{R^2}\right) 2\pi r\, dr = \pi R^2 V_{\text{avg}} = \frac{\pi R^4}{8\mu}\left(-\frac{dp}{dz}\right) = \frac{\pi R^4 \Delta p}{8\mu L}$$

$$\tau_{\text{wall}} = \mu\left|\frac{\partial \upsilon_z}{\partial r}\right|_{r=R} = \frac{4\mu V_{\text{avg}}}{R} = \frac{R}{2}\left(-\frac{dp}{dz}\right) = \frac{R}{2}\frac{\Delta p}{L} \tag{4.138}$$

Note that we have substituted the equality $(-dp/dz) = \Delta p/L$, where Δp is the pressure drop along the entire length L of the pipe.

These formulas are valid as long as the flow is *laminar*—that is, when the dimensionless Reynolds number of the flow, $\mathrm{Re}_D = \rho V_{avg}(2R)/\mu$, is less than about 2100. Note also that the formulas do not depend on density, the reason being that the convective acceleration of this flow is zero.

EXAMPLE 4.11

SAE 10W oil at 20°C flows at 1.1 m^3/h through a horizontal pipe with d = 2 cm and L = 12 m. Find (*a*) the average velocity, (*b*) the Reynolds number, (*c*) the pressure drop, and (*d*) the power required.

Solution

- *Assumptions:* Laminar, steady, Hagen-Poiseuille pipe flow.
- *Approach:* The formulas of Eqs. (4.138) are appropriate for this problem. Note that R = 0.01 m.
- *Property values:* From Table A.3 for SAE 10W oil, ρ = 870 kg/m^3 and μ = 0.104 kg/(m-s).
- *Solution steps:* The average velocity follows easily from the flow rate and the pipe area:

$$V_{avg} = \frac{Q}{\pi R^2} = \frac{(1.1/3600)\ \mathrm{m^3/s}}{\pi(0.01\ \mathrm{m})^2} = 0.973\ \frac{\mathrm{m}}{\mathrm{s}} \qquad \textit{Ans. (a)}$$

We had to convert Q to m^3/s. The (diameter) Reynolds number follows from the average velocity:

$$\mathrm{Re}_d = \frac{\rho V_{avg} d}{\mu} = \frac{(870\ \mathrm{kg/m^3})(0.973\ \mathrm{m/s})(0.02\ \mathrm{m})}{0.104\ \mathrm{kg/(m\text{-}s)}} = 163 \qquad \textit{Ans. (b)}$$

This is less than the "transition" value of 2100; so the flow is indeed *laminar*, and the formulas are valid. The pressure drop is computed from the third of Eqs. (4.138):

$$Q = \frac{1.1}{3600}\frac{\mathrm{m^3}}{\mathrm{s}} = \frac{\pi R^4 \Delta p}{8\mu L} = \frac{\pi(0.01\ \mathrm{m})^4 \Delta p}{8(0.104\ \mathrm{kg/(m\text{-}s)})(12\ \mathrm{m})} \quad \text{solve for } \Delta p = 97{,}100\ \mathrm{Pa} \qquad \textit{Ans. (c)}$$

When using SI units, the answer returns in pascals; no conversion factors are needed. Finally, the power required is the product of flow rate and pressure drop:

$$\text{Power} = Q\Delta p = \left(\frac{1.1}{3600}\ \mathrm{m^3/s}\right)(97{,}100\ \mathrm{N/m^2}) = 29.7\ \frac{\mathrm{N\text{-}m}}{\mathrm{s}} = 29.7\ \mathrm{W} \qquad \textit{Ans. (d)}$$

- *Comments:* Pipe flow problems are straightforward algebraic exercises if the data are compatible. Note again that SI units can be used in the formulas without conversion factors.

Flow between Long Concentric Cylinders

Consider a fluid of constant (ρ, μ) between two concentric cylinders, as in Fig. 4.13. There is no axial motion or end effect $\upsilon_z = \partial/\partial z = 0$. Let the inner cylinder rotate at angular velocity Ω_i. Let the outer cylinder be fixed. There is circular symmetry, so the velocity does not vary with θ and varies only with r.

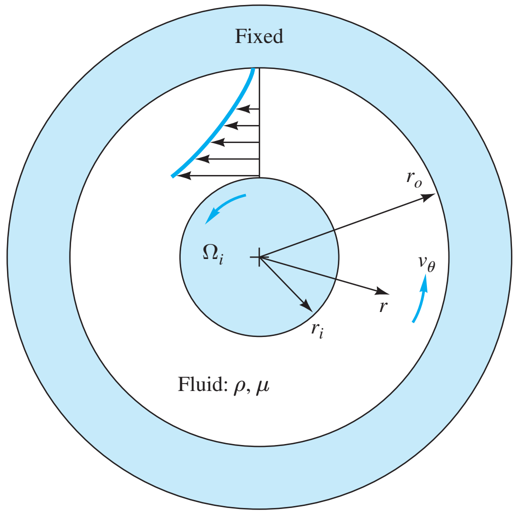


Fig. 4.13 Coordinate system for incompressible viscous flow between a fixed outer cylinder and a steadily rotating inner cylinder.

The continuity equation for this problem is Eq. (4.12*b*) with $\upsilon_z = 0$:

$$\frac{1}{r}\frac{\partial}{\partial r}(r\upsilon_r) + \frac{1}{r}\frac{\partial \upsilon_\theta}{\partial \theta} = 0 = \frac{1}{r}\frac{d}{dr}(r\upsilon_r) \quad \text{or} \quad r\upsilon_r = \text{const}$$

Note that υ_θ does not vary with θ. Since $\upsilon_r = 0$ at both the inner and outer cylinders, it follows that $\upsilon_r = 0$ everywhere and the motion can only be purely circumferential, $\upsilon_\theta = \upsilon_\theta(r)$. The θ-momentum equation (D.6) becomes

$$\rho(\mathbf{V}\cdot\boldsymbol{\nabla})\upsilon_\theta + \frac{\rho\upsilon_r\upsilon_\theta}{r} = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \rho g_\theta + \mu\left(\nabla^2\upsilon_\theta - \frac{\upsilon_\theta}{r^2}\right)$$

For the conditions of the present problem, all terms are zero except the last. Therefore, the basic differential equation for flow between rotating cylinders is

$$\nabla^2\upsilon_\theta = \frac{1}{r}\frac{d}{dr}\left(r\frac{d\upsilon_\theta}{dr}\right) = \frac{\upsilon_\theta}{r^2} \tag{4.139}$$

This is a linear second-order ordinary differential equation with the solution

$$\upsilon_\theta = C_1 r + \frac{C_2}{r}$$

The constants are found by the no-slip condition at the inner and outer cylinders:

Outer, at $r = r_o$: $$\upsilon_\theta = 0 = C_1 r_o + \frac{C_2}{r_o}$$

Inner, at $r = r_i$: $$\upsilon_\theta = \Omega_i r_i = C_1 r_i + \frac{C_2}{r_i}$$

The final solution for the velocity distribution is

Rotating inner cylinder: $$\upsilon_\theta = \Omega_i r_i \frac{r_o/r - r/r_o}{r_o/r_i - r_i/r_o} \tag{4.140}$$

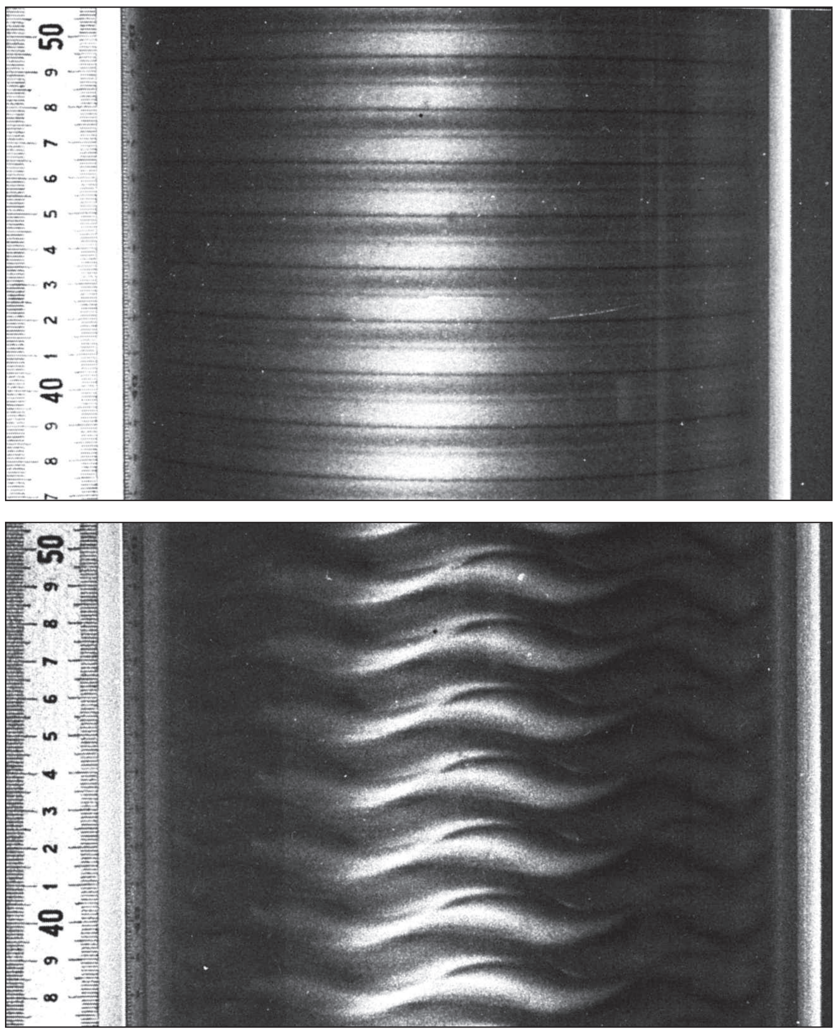
The velocity profile closely resembles the sketch in Fig. 4.13. Variations of this case, such as a rotating outer cylinder, are given in the problem assignments.

Instability of Rotating Inner[15] Cylinder Flow

The classic *Couette flow* solution[16] of Eq. (4.140) describes a physically satisfying concave, two-dimensional, laminar flow velocity profile as in Fig. 4.13. The solution is mathematically exact for an incompressible fluid. However, it becomes unstable at a relatively low rate of rotation of the inner cylinder, as shown in 1923 in a classic paper by G. I. Taylor [17]. At a critical value of what is now called the dimensionless *Taylor number,* denoted Ta,

$$\mathrm{Ta}_{\mathrm{crit}} = \frac{r_i(r_o - r_i)^3\Omega_i^2}{\nu^2} \approx 1700 \tag{4.141}$$

the plane flow of Fig. 4.13 vanishes and is replaced by a laminar *three-dimensional* flow pattern consisting of rows of nearly square alternating toroidal vortices.



(*a*)

(*b*)

Fig. 4.14 Experimental verification of the instability of flow between a fixed outer and a rotating inner cylinder. (*a*) Toroidal Taylor vortices exist at 1.16 times the critical speed; (*b*) at 8.5 times the critical speed, the vortices are doubly periodic. (*Courtesy of Cambridge University Press—E.L. Koschmieder, "Turbulent Taylor Vortex Flow," Journal of Fluid Mechanics, vol. 93. pt. 3, 1979, pp. 515–527.*) This instability does not occur if only the outer cylinder rotates.

[15]This section may be omitted without loss of continuity.

[16]Named after M. Couette, whose pioneering paper in 1890 established rotating cylinders as a method, still used today, for measuring the viscosity of fluids.

An experimental demonstration of toroidal "Taylor vortices" is shown in Fig. 4.14*a*, measured at Ta $\approx$ 1.16 Ta_{crit} by Koschmieder [18]. At higher Taylor numbers, the vortices also develop a circumferential periodicity but are still laminar, as illustrated in Fig. 4.14*b*. At still higher Ta, turbulence ensues. This interesting instability reminds us that the Navier-Stokes equations, being nonlinear, do admit to multiple (nonunique) laminar solutions in addition to the usual instabilities associated with turbulence and chaotic dynamic systems.

Summary

This chapter complements Chap. 3 by using an infinitesimal control volume to derive the basic partial differential equations of mass, momentum, and energy for a fluid. These equations, together with thermodynamic state relations for the fluid and appropriate boundary conditions, in principle can be solved for the complete flow field in any given fluid mechanics problem. Except for Chap. 9, in most of the problems to be studied here an incompressible fluid with constant viscosity is assumed.

In addition to deriving the basic equations of mass, momentum, and energy, this chapter introduced some supplementary ideas—the stream function, vorticity, irrotationality, and the velocity potential—which will be useful in coming chapters, especially Chap. 8. Temperature and density variations will be neglected except in Chap. 9, where compressibility is studied.

This chapter ended by discussing a few classic solutions for laminar viscous flows (Couette flow due to moving walls, Poiseuille duct flow due to pressure gradient, and flow between rotating cylinders). Whole books [4, 5, 9–11, 15] discuss classic approaches to fluid mechanics, and other texts [6, 12–14] extend these studies to the realm of continuum mechanics. This does not mean that all problems can be solved analytically. The new field of computational fluid dynamics [1] shows great promise of achieving approximate solutions to a wide variety of flow problems. In addition, when the geometry and boundary conditions are truly complex, experimentation (Chap. 5) is a preferred alternative.

Problems

Most of the problems herein are fairly straightforward. More difficult or open-ended assignments are labeled with an asterisk. Problems labeled with a computer icon may require the use of a computer. The standard end-of-chapter problems P4.1 to P4.99 (categorized in the problem list here) are followed by word problems W4.1 to W4.10, fundamentals of engineering exam problems FE4.1 to FE4.6, and comprehensive problems C4.1 and C4.2.

Problem Distribution

Section	Topic	Problems
4.1	The acceleration of a fluid	P4.1–P4.8
4.2	The continuity equation	P4.9–P4.25
4.3	Linear momentum: Navier-Stokes	P4.26–P4.38
4.4	Angular momentum: couple stresses	P4.39
4.5	The differential energy equation	P4.40–P4.41
4.6	Boundary conditions	P4.42–P4.46
4.7	Stream function	P4.47–P4.55
4.8 and 4.9	Velocity potential, vorticity	P4.56–P4.67
4.7 and 4.9	Stream function and velocity potential	P4.68–P4.78
4.10	Incompressible viscous flows	P4.79–P4.96
4.10	Slip flows	P4.97–P4.99

The acceleration of a fluid

P4.1 An idealized velocity field is given by the formula

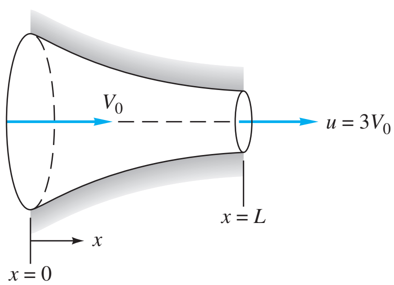
$$\mathbf{V} = 4tx\mathbf{i} - 2t^2y\mathbf{j} + 4xz\mathbf{k}$$

Is this flow field steady or unsteady? Is it two- or three-dimensional? At the point $(x, y, z) = (-1, 1, 0)$, compute (*a*) the acceleration vector and (*b*) any unit vector normal to the acceleration.

P4.2 Flow through the converging nozzle in Fig. P4.2 can be approximated by the one-dimensional velocity distribution

$$u \approx V_0\left(1 + \frac{2x}{L}\right) \quad \upsilon \approx 0 \quad w \approx 0$$

(a) Find a general expression for the fluid acceleration in the nozzle. (b) For the specific case $V_0 = 10$ ft/s and $L = 6$ in, compute the acceleration, in g's, at the entrance and at the exit.



P4.2

P4.3 A two-dimensional velocity field is given by

$$\mathbf{V} = (x^2 - y^2 + x)\mathbf{i} - (2xy + y)\mathbf{j}$$

in arbitrary units. At $(x, y) = (1, 2)$, compute (a) the accelerations a_x and a_y, (b) the velocity component in the direction $\theta = 40°$, (c) the direction of maximum velocity, and (d) the direction of maximum acceleration.

P4.4 A simple flow model for a two-dimensional converging nozzle is the distribution

$$u = U_0\left(1 + \frac{x}{L}\right) \quad v = -U_0\frac{y}{L} \quad w = 0$$

(a) Sketch a few streamlines in the region $0 < x/L < 1$ and $0 < y/L < 1$, using the method of Sec. 1.11. (b) Find expressions for the horizontal and vertical accelerations. (c) Where is the largest resultant acceleration and its numerical value?

P4.5 The velocity field near a stagnation point may be written in the form

$$u = \frac{U_0 x}{L} \quad \upsilon = -\frac{U_0 y}{L} \quad U_0 \text{ and } L \text{ are constants}$$

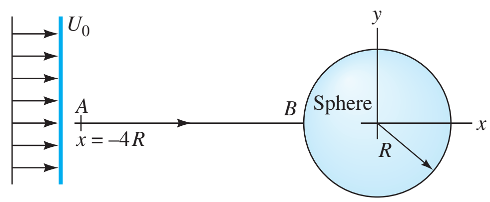
(a) Show that the acceleration vector is purely radial. (b) For the particular case $L = 1.5$ m, if the acceleration at $(x, y) = (1\text{ m}, 1\text{ m})$ is 25 m/s^2, what is the value of U_0?

P4.6 In deriving the continuity equation, we assumed, for simplicity, that the mass flow per unit area on the left face was just ρu. In fact, ρu varies also with y and z, and thus it must be different on the four corners of the left face. Account for these variations, average the four corners, and determine how this might change the inlet mass flow from $\rho u\, dy\, dz$.

P4.7 Consider a sphere of radius R immersed in a uniform stream U_0, as shown in Fig. P4.7. According to the theory of Chap. 8, the fluid velocity along streamline AB is given by

$$\mathbf{V} = u\mathbf{i} = U_0\left(1 + \frac{R^3}{x^3}\right)\mathbf{i}$$

Find (a) the position of maximum fluid acceleration along AB and (b) the time required for a fluid particle to travel from A to B.

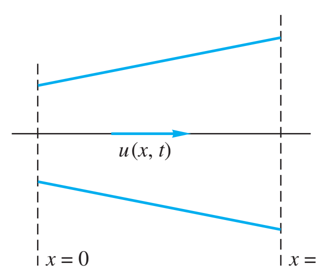


P4.7

P4.8 When a valve is opened, fluid flows in the expansion duct of Fig. P4.8 according to the approximation

$$\mathbf{V} = \mathbf{i}U\left(1 - \frac{x}{2L}\right)\tanh\frac{Ut}{L}$$

Find (a) the fluid acceleration at $(x, t) = (L, L/U)$ and (b) the time for which the fluid acceleration at $x = L$ is zero. Why does the fluid acceleration become negative after condition (b)?



P4.8

The continuity equation

P4.9 An idealized incompressible flow has the proposed three-dimensional velocity distribution

$$\mathbf{V} = 4xy^2\mathbf{i} + f(y)\mathbf{j} - zy^2\mathbf{k}$$

Find the appropriate form of the function $f(y)$ that satisfies the continuity relation.

P4.10 A two-dimensional, incompressible flow has the velocity components $u = 4y$ and $v = 2x$. (*a*) Find the acceleration components. (*b*) Is the vector acceleration radial? (*c*) Sketch a few streamlines in the first quadrant and determine if any are straight lines.

P4.11 Derive Eq. (4.12*b*) for cylindrical coordinates by considering the flux of an incompressible fluid in and out of the elemental control volume in Fig. 4.2.

P4.12 Spherical polar coordinates (r, θ, ϕ) are defined in Fig. P4.12. The cartesian transformations are

$$x = r \sin\theta \cos\phi$$
$$y = r \sin\theta \sin\phi$$
$$z = r \cos\theta$$

Do not show that the cartesian incompressible continuity relation [Eq. (4.12*a*)] can be transformed to the spherical polar form

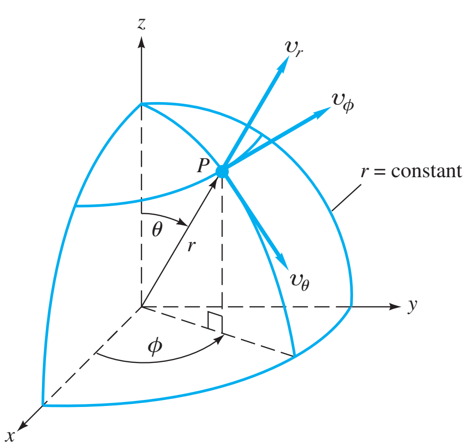
$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \upsilon_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\upsilon_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(\upsilon_\phi) = 0$$

What is the most general form of υ_r when the flow is purely radial—that is, υ_θ and υ_ϕ are zero?

P4.13 For an incompressible plane flow in polar coordinates, we are given

$$\upsilon_r = r^3 \cos\theta + r^2 \sin\theta$$

Find the appropriate form of circumferential velocity for which continuity is satisfied.



P4.12

P4.14 For incompressible polar coordinate flow, what is the most general form of a purely circulatory motion, $\upsilon_\theta = \upsilon_\theta(r, \theta, t)$ and $\upsilon_r = 0$, that satisfies continuity?

P4.15 What is the most general form of a purely radial polar coordinate incompressible flow pattern, $\upsilon_r = \upsilon_r(r, \theta, t)$ and $\upsilon_\theta = 0$, that satisfies continuity?

P4.16 Consider the plane polar coordinate velocity distribution

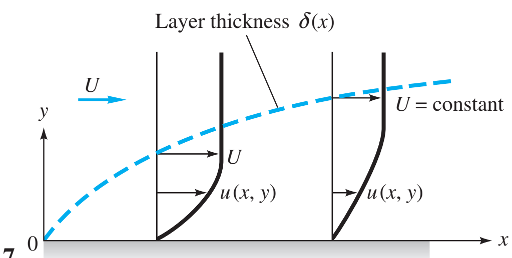
$$v_r = \frac{C}{r} \qquad v_\theta = \frac{K}{r} \qquad v_z = 0$$

where C and K are constants. (*a*) Determine if the equation of continuity is satisfied. (*b*) By sketching some velocity vector directions, plot a single streamline for $C = K$. What might this flow field simulate?

P4.17 An excellent approximation for the two-dimensional incompressible laminar boundary layer on the flat surface in Fig. P4.17 is

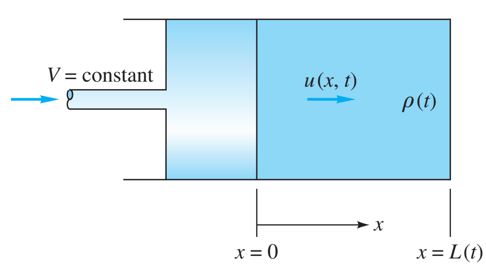
$$u \approx U\left(2\frac{y}{\delta} - 2\frac{y^3}{\delta^3} + \frac{y^4}{\delta^4}\right) \qquad \text{for } y \le \delta$$
$$\text{where } \delta = Cx^{1/2},\ C = \text{const}$$

(*a*) Assuming a no-slip condition at the wall, find an expression for the velocity component $\upsilon(x, y)$ for $y \le \delta$. (*b*) Then find the maximum value of υ at the station $x = 1$ m, for the particular case of airflow, when $U = 3$ m/s and $\delta = 1.1$ cm.



P4.17

P4.18 A piston compresses gas in a cylinder by moving at constant speed V, as in Fig. P4.18. Let the gas density and length at $t = 0$ be ρ_0 and L_0, respectively. Let the gas velocity vary linearly from $u = V$ at the piston face to $u = 0$ at $x = L$. If the gas density varies only with time, find an expression for $\rho(t)$.



P4.18

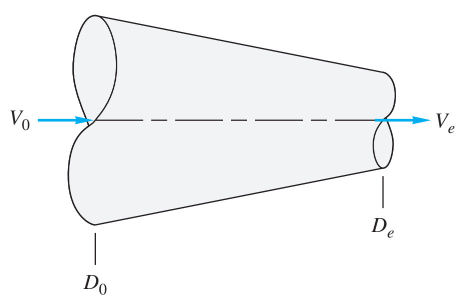
P4.19 A proposed incompressible plane flow in polar coordinates is given by

$$v_r = 2r\cos(2\theta); \quad v_\theta = -2r\sin(2\theta)$$

(a) Determine if this flow satisfies the equation of continuity. (b) If so, sketch a possible streamline in the first quadrant by finding the velocity vectors at $(r, \theta) = (1.25, 20°)$, $(1.0, 45°)$, and $(1.25, 70°)$. (c) Speculate on what this flow might represent.

P4.20 A two-dimensional incompressible velocity field has $u = K(1 - e^{-ay})$, for $x \le L$ and $0 \le y \le \infty$. What is the most general form of $\upsilon(x, y)$ for which continuity is satisfied and $\upsilon = \upsilon_0$ at $y = 0$? What are the proper dimensions for constants K and a?

P4.21 Air flows under steady, approximately one-dimensional conditions through the conical nozzle in Fig. P4.21. If the speed of sound is approximately 340 m/s, what is the minimum nozzle-diameter ratio D_e/D_0 for which we can safely neglect compressibility effects if $V_0 =$ (a) 10 m/s and (b) 30 m/s?



P4.21

P4.22 In an *axisymmetric* flow, nothing varies with θ, and the only nonzero velocities are υ_r and υ_z (see Fig. 4.2). If the flow is steady and incompressible and $\upsilon_z = Bz$, where B is constant, find the most general form of υ_r which satisfies continuity.

P4.23 A tank volume $\mathscr{V}$ contains gas at conditions (ρ_0, p_0, T_0). At time $t = 0$ it is punctured by a small hole of area A. According to the theory of Chap. 9, the mass flow out of such a hole is approximately proportional to A and to the tank pressure. If the tank temperature is assumed constant and the gas is ideal, find an expression for the variation of density within the tank.

P4.24 For laminar flow between parallel plates (see Fig. 4.12b), the flow is two-dimensional $(\upsilon \ne 0)$ if the walls are porous. A special case solution is $u = (A - Bx)(h^2 - y^2)$, where A and B are constants. (a) Find a general formula for velocity υ if $\upsilon = 0$ at $y = 0$. (b) What is the value of the constant B if $\upsilon = \upsilon_w$ at $y = +h$?

P4.25 An incompressible flow in polar coordinates is given by

$$\upsilon_r = K\cos\theta\left(1 - \frac{b}{r^2}\right)$$

$$\upsilon_\theta = -K\sin\theta\left(1 + \frac{b}{r^2}\right)$$

Does this field satisfy continuity? For consistency, what should the dimensions of constants K and b be? Sketch the surface where $\upsilon_r = 0$ and interpret.

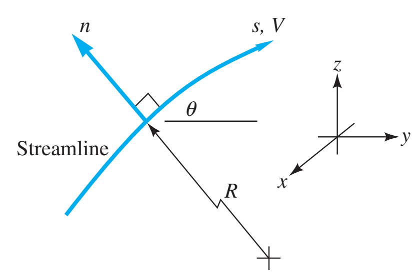
Linear momentum: Navier-Stokes

***P4.26** Curvilinear, or streamline, coordinates are defined in Fig. P4.26, where n is normal to the streamline in the plane of the radius of curvature R. Euler's frictionless momentum equation (4.36) in streamline coordinates becomes

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial s} = -\frac{1}{\rho}\frac{\partial p}{\partial s} + g_s \qquad (1)$$

$$-V\frac{\partial \theta}{\partial t} - \frac{V^2}{R} = -\frac{1}{\rho}\frac{\partial p}{\partial n} + g_n \qquad (2)$$

Show that the integral of Eq. (1) with respect to s is none other than our old friend Bernoulli's equation (3.54).



P4.26

P4.27 A frictionless, incompressible steady flow field is given by

$$\mathbf{V} = 2xy\mathbf{i} - y^2\mathbf{j}$$

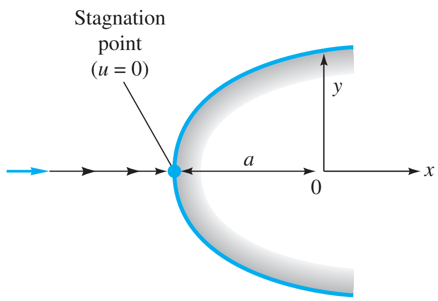
in arbitrary units. Let the density be ρ_0 = constant and neglect gravity. Find an expression for the pressure gradient in the x direction.

P4.28 For the velocity distribution of Prob. 4.10, (a) check continuity. (b) Are the Navier-Stokes equations valid? (c) If so, determine $p(x, y)$ if the pressure at the origin is p_0.

P4.29 Consider a steady, two-dimensional, incompressible flow of a newtonian fluid in which the velocity field is known: $u = -2xy$, $\upsilon = y^2 - x^2$, $w = 0$. (a) Does this flow satisfy conservation of mass? (b) Find the pressure field, $p(x, y)$ if the pressure at the point $(x = 0, y = 0)$ is equal to p_a.

P4.30 For the velocity distribution of Prob. P4.4, determine if (*a*) the equation of continuity and (*b*) the Navier-Stokes equation are satisfied. (*c*) If the latter is true, find the pressure distribution $p(x, y)$ when the pressure at the origin equals p_o.

P4.31 According to potential theory (Chap. 8) for the flow approaching a rounded two-dimensional body, as in Fig. P4.31, the velocity approaching the stagnation point is given by $u = U(1 - a^2/x^2)$, where a is the nose radius and U is the velocity far upstream. Compute the value and position of the maximum viscous normal stress along this streamline.

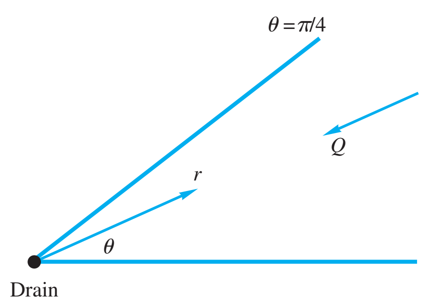


P4.31

Is this also the position of maximum fluid deceleration? Evaluate the maximum viscous normal stress if the fluid is SAE 30 oil at 20°C, with $U = 2$ m/s and $a = 6$ cm.

P4.32 The answer to Prob. P4.14 is $\upsilon_\theta = f(r)$ only. Do not reveal this to your friends if they are still working on Prob. P4.14. Show that this flow field is an exact solution to the Navier-Stokes equations (4.38) for only two special cases of the function $f(r)$. Neglect gravity. Interpret these two cases physically.

P4.33 Consider incompressible flow at a volume rate Q toward a drain at the vertex of a 45° wedge of width b, as in Fig. P4.33. Neglect gravity and friction and assume purely radial inflow. (*a*) Find an expression for $\upsilon_r(r)$. (*b*) Show that the viscous term in the r-momentum equation is zero. (*c*) Find the pressure distribution $p(r)$ if $p = p_o$ at $r = R$.



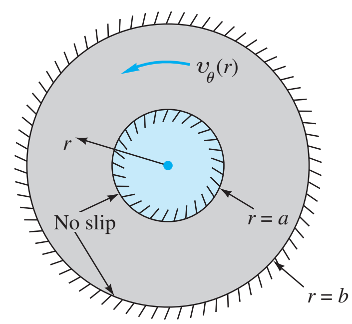
P4.33

P4.34 A proposed three-dimensional incompressible flow field has the following vector form:

$$\mathbf{V} = Kx\mathbf{i} + Ky\mathbf{j} - 2Kz\mathbf{k}$$

(*a*) Determine if this field is a valid solution to continuity and Navier-Stokes. (*b*) If $\mathbf{g} = -g\mathbf{k}$, find the pressure field $p(x, y, z)$. (*c*) Is the flow irrotational?

P4.35 From the Navier-Stokes equations for incompressible flow in polar coordinates (App. D for cylindrical coordinates), find the most general case of purely circulating motion $\upsilon_\theta(r)$, $\upsilon_r = \upsilon_z = 0$, for flow with no slip between two fixed concentric cylinders, as in Fig. P4.35.

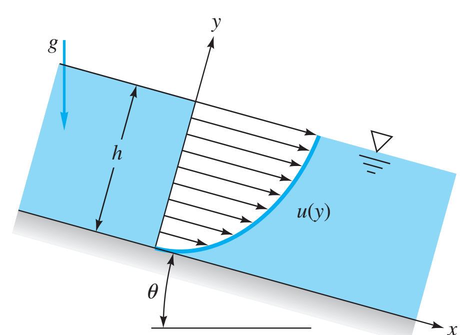


P4.35

P4.36 A constant-thickness film of viscous liquid flows in laminar motion down a plate inclined at angle θ, as in Fig. P4.36. The velocity profile is

$$u = Cy(2h - y) \quad \upsilon = w = 0$$

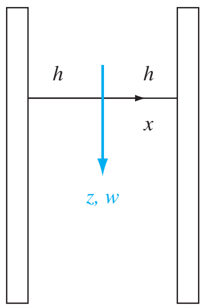
Find the constant C in terms of the specific weight and viscosity and the angle θ. Find the volume flux Q per unit width in terms of these parameters.



P4.36

***P4.37** A viscous liquid of constant ρ and μ falls due to gravity between two plates a distance $2h$ apart, as in Fig. P4.37. The flow is fully developed, with a single velocity component

$w = w(x)$. There are no applied pressure gradients, only gravity. Solve the Navier-Stokes equation for the velocity profile between the plates.



P4.37

P4.38 Show that the incompressible flow distribution, in cylindrical coordinates,

$$v_r = 0 \qquad v_\theta = Cr^n \qquad v_z = 0$$

where C is a constant, (*a*) satisfies the Navier-Stokes equation for only two values of n. Neglect gravity. (*b*) Knowing that $p = p(r)$ only, find the pressure distribution for each case, assuming that the pressure at $r = R$ is p_0. What might these two cases represent?

Angular momentum: couple stresses

P4.39 Reconsider the angular momentum balance of Fig. 4.5 by adding a concentrated *body couple* C_z about the z axis [6]. Determine a relation between the body couple and shear stress for equilibrium. What are the proper dimensions for C_z? (Body couples are important in continuous media with microstructure, such as granular materials.)

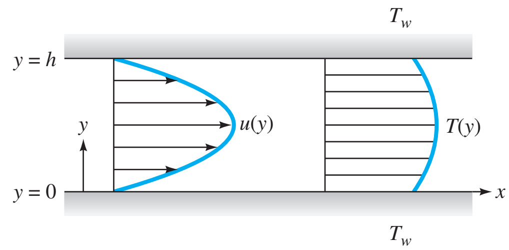
The differential energy equation

P4.40 For pressure-driven laminar flow between parallel plates (see Fig. 4.12*b*), the velocity components are $u = U(1-y^2/h^2)$, $\upsilon = 0$, and $w = 0$, where U is the centerline velocity. In the spirit of Ex. 4.6, find the temperature distribution $T(y)$ for a constant wall temperature T_w.

P4.41 As mentioned in Sec. 4.10, the velocity profile for laminar flow between two plates, as in Fig. P4.41, is

$$u = \frac{4u_{max}y(h-y)}{h^2} \qquad \upsilon = w = 0$$

If the wall temperature is T_w at both walls, use the incompressible flow energy equation (4.75) to solve for the temperature distribution $T(y)$ between the walls for steady flow.



P4.41

Boundary conditions

P4.42 Suppose we wish to analyze the rotating, partly full cylinder of Fig. 2.23 as a *spin-up* problem, starting from rest and continuing until solid-body rotation is achieved. What are the appropriate boundary and initial conditions for this problem?

P4.43 For the draining liquid film of Fig. P4.36, what are the appropriate boundary conditions (*a*) at the bottom $y = 0$ and (*b*) at the surface $y = h$?

P4.44 Suppose that we wish to analyze the sudden pipe expansion flow of Fig. P3.59, using the full continuity and Navier-Stokes equations. What are the proper boundary conditions to handle this problem?

P4.45 For the sluice gate problem of Example 3.10, list all the boundary conditions needed to solve this flow exactly by, say, computational fluid dynamics.

P4.46 Fluid from a large reservoir at temperature T_0 flows into a circular pipe of radius R. The pipe walls are wound with an electric resistance coil that delivers heat to the fluid at a rate q_w (energy per unit wall area). If we wish to analyze this problem by using the full continuity, Navier-Stokes, and energy equations, what are the proper boundary conditions for the analysis?

Stream function

P4.47 A two-dimensional incompressible flow is given by the velocity field $\mathbf{V} = 3y\mathbf{i} + 2x\mathbf{j}$, in arbitrary units. Does this flow satisfy continuity? If so, find the stream function $\psi(x, y)$ and plot a few streamlines, with arrows.

P4.48 Consider the following two-dimensional incompressible flow, which clearly satisfies continuity:

$$u = U_0 = \text{constant}, \ \upsilon = V_0 = \text{constant}$$

Find the stream function $\psi(r, \theta)$ of this flow using *polar coordinates.*

P4.49 Investigate the stream function $\psi = K(x^2 - y^2)$, $K =$ constant. Plot the streamlines in the full xy plane, find any stagnation points, and interpret what the flow could represent.

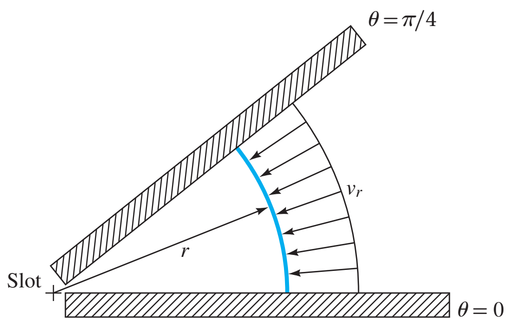
P4.50 In 1851, George Stokes (of Navier-Stokes fame) solved the problem of steady incompressible low-Reynolds-number flow past a sphere, using *spherical polar* coordinates (r, θ) [Ref. 5, page 168]. In these coordinates, the equation of continuity is

$$\frac{\partial}{\partial r}(r^2 \upsilon_r \sin\theta) + \frac{\partial}{\partial \theta}(r \upsilon_\theta \sin\theta) = 0$$

(*a*) Does a stream function exist for these coordinates? (*b*) If so, find its form.

P4.51 The velocity profile for pressure-driven laminar flow between parallel plates (see Fig. 4.12*b*) has the form $u = C(h^2 - y^2)$, where C is a constant. (*a*) Determine if a stream function exists. (*b*) If so, find a formula for the stream function.

P4.52 A two-dimensional, incompressible, frictionless fluid is guided by wedge-shaped walls into a small slot at the origin, as in Fig. P4.52. The width into the paper is b,



P4.52

and the volume flow rate is Q. At any given distance r from the slot, the flow is radial inward, with constant velocity. Find an expression for the polar coordinate stream function of this flow.

P4.53 For the fully developed laminar pipe flow solution of Eq. (4.137), find the axisymmetric stream function $\psi(r, z)$. Use this result to determine the average velocity $V = Q/A$ in the pipe as a ratio of u_{max}.

P4.54 An incompressible stream function is defined by

$$\psi(x, y) = \frac{U}{L^2}(3x^2y - y^3)$$

where U and L are (positive) constants. Where in this chapter are the streamlines of this flow plotted? Use this stream function to find the volume flow Q passing through the rectangular surface whose corners are defined by $(x, y, z) = (2L, 0, 0)$, $(2L, 0, b)$, $(0, L, b)$, and $(0, L, 0)$. Show the direction of Q.

P4.55 The proposed flow in Prob. P4.19 does indeed satisfy the equation of continuity. Determine the polar-coordinate stream function of this flow.

Velocity potential, vorticity

P4.56 Investigate the velocity potential $\phi = Kxy$, K = constant. Sketch the potential lines in the full xy plane, find any stagnation points, and sketch in by eye the orthogonal streamlines. What could the flow represent?

P4.57 A two-dimensional incompressible flow field is defined by the velocity components

$$u = 2V\left(\frac{x}{L} - \frac{y}{L}\right) \qquad \upsilon = -2V\frac{y}{L}$$

where V and L are constants. If they exist, find the stream function and velocity potential.

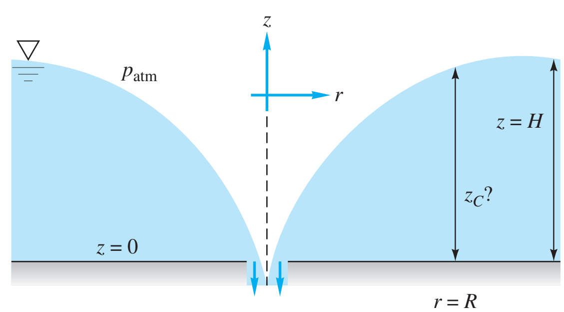
P4.58 Show that the incompressible velocity potential in plane polar coordinates $\phi(r, \theta)$ is such that

$$\upsilon_r = \frac{\partial \phi}{\partial r} \quad \upsilon_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

Finally show that ϕ as defined here satisfies Laplace's equation in polar coordinates for incompressible flow.

P4.59 Consider the two-dimensional incompressible velocity potential $\phi = xy + x^2 - y^2$. (*a*) Is it true that $\nabla^2\phi = 0$, and, if so, what does this mean? (*b*) If it exists, find the stream function $\psi(x, y)$ of this flow. (*c*) Find the equation of the streamline that passes through $(x, y) = (2, 1)$.

P4.60 Liquid drains from a small hole in a tank, as shown in Fig. P4.60, such that the velocity field set up is given by $\upsilon_r \approx 0$, $\upsilon_z \approx 0$, $\upsilon_\theta = KR^2/r$, where $z = H$ is the depth of the water far from the hole. Is this flow pattern rotational or irrotational? Find the depth z_C of the water at the radius $r = R$.



P4.60

P4.61 An incompressible stream function is given by $\psi = a\theta + br \sin\theta$. (*a*) Does this flow have a velocity potential? (*b*) If so, find it.

P4.62 Show that the linear Couette flow between plates in Fig. 1.7 has a stream function but no velocity potential. Why is this so?

P4.63 Find the two-dimensional velocity potential $\phi(r, \theta)$ for the polar coordinate flow pattern $\upsilon_r = Q/r$, $\upsilon_\theta = K/r$, where Q and K are constants.

P4.64 Show that the velocity potential $\phi(r, z)$ in axisymmetric cylindrical coordinates (see Fig. 4.2) is defined such that

$$\upsilon_r = \frac{\partial \phi}{\partial r} \qquad \upsilon_z = \frac{\partial \phi}{\partial z}$$

Further show that for incompressible flow this potential satisfies Laplace's equation in (r, z) coordinates.

P4.65 Consider the function $f = ay - by^3$. (*a*) Could this represent a realistic velocity potential? *Extra credit*: (*b*) Could it represent a stream function?

P4.66 A plane polar coordinate velocity potential is defined by

$$\phi = \frac{K \cos\theta}{r} \qquad K = \text{const}$$

Find the stream function for this flow, sketch some streamlines and potential lines, and interpret the flow pattern.

P4.67 A stream function for a plane, irrotational, polar coordinate flow is

$$\psi = C\theta - K \ln r \qquad C \text{ and } K = \text{const}$$

Find the velocity potential for this flow. Sketch some streamlines and potential lines, and interpret the flow pattern.

Stream function *and* velocity potential

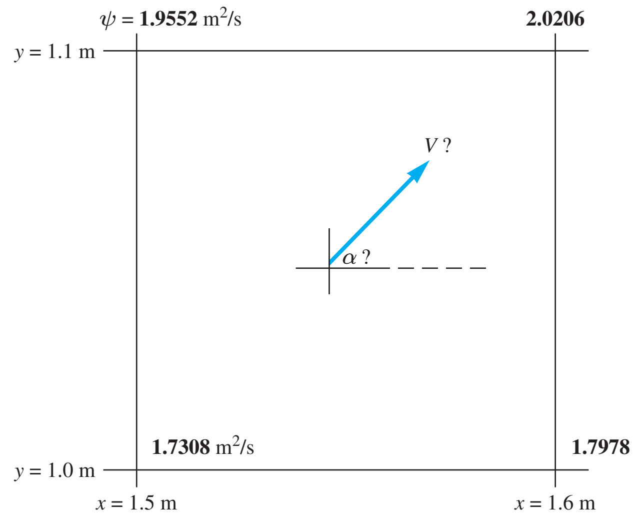
P4.68 For the velocity distribution of Prob. P4.4, (*a*) determine if a velocity potential exists, and (*b*), if it does, find an expression for $\phi(x, y)$ and sketch the potential line which passes through the point $(x, y) = (L/2, L/2)$.

P4.69 A steady, two-dimensional flow has the following polar-coordinate velocity potential:

$$\phi = C\, r \cos\theta + K \ln r$$

where C and K are constants. Determine the stream function $\psi(r, \theta)$ for this flow. For extra credit, let C be a velocity scale U and let $K = UL$, sketch what the flow might represent.

P4.70 A CFD model of steady two-dimensional incompressible flow has printed out the values of stream function $\psi(x, y)$, in m^2/s, at each of the four corners of a small 10-cm-by-10-cm cell, as shown in Fig. P4.70. Use these numbers to estimate the resultant velocity in the center of the cell and its angle α with respect to the x axis.



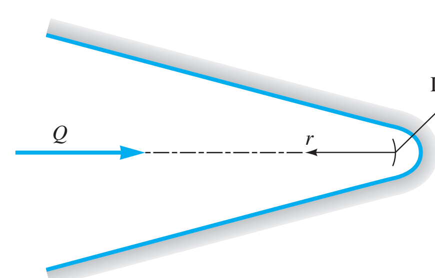
P4.70

P4.71 Consider the following two-dimensional function $f(x, y)$:

$$f = Ax^3 + Bxy^2 + Cx^2 + D \quad \text{where } A > 0$$

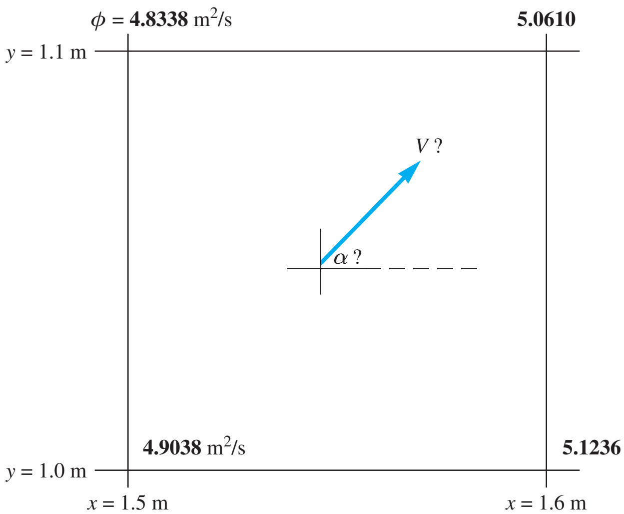
(*a*) Under what conditions, if any, on (A, B, C, D) can this function be a steady plane-flow velocity potential? (*b*) If you find a $\phi(x, y)$ to satisfy part (*a*), also find the associated stream function $\psi(x, y)$, if any, for this flow.

P4.72 Water flows through a two-dimensional narrowing wedge at 9.96 gal/min per meter of width into the paper (Fig. P4.72). If this inward flow is purely radial, find an expression, in SI units, for (*a*) the stream function and (*b*) the velocity potential of the flow. Assume one-dimensional flow. The included angle of the wedge is 45°.



P4.72

P4.73 A CFD model of steady two-dimensional incompressible flow has printed out the values of velocity potential $\phi(x, y)$, in m^2/s, at each of the four corners of a small 10-cm-by-10-cm cell, as shown in Fig. P4.73. Use these numbers to estimate the resultant velocity in the center of the cell and its angle α with respect to the x axis.



P4.73

P4.74 Consider the two-dimensional incompressible polar-coordinate velocity potential

$$\phi = Br\cos\theta + B\,L\,\theta$$

where B is a constant and L is a constant length scale. (*a*) What are the dimensions of B? (*b*) Locate the only stagnation point in this flow field. (*c*) Prove that a stream function exists and then find the function $\psi(r, \theta)$.

P4.75 Given the following steady *axisymmetric* stream function:

$$\psi = \frac{B}{2}\left(r^2 - \frac{r^4}{2R^2}\right) \quad \text{where } B \text{ and } R \text{ are constants}$$

valid in the region $0 \le r \le R$ and $0 \le z \le L$. (*a*) What are the dimensions of the constant B? (*b*) Show whether this flow possesses a velocity potential, and, if so, find it. (*c*) What might this flow represent? *Hint:* Examine the axial velocity v_z.

***P4.76** A two-dimensional incompressible flow has the velocity potential

$$\phi = K(x^2 - y^2) + C\ln(x^2 + y^2)$$

where K and C are constants. In this discussion, avoid the origin, which is a singularity (infinite velocity). (*a*) Find the sole stagnation point of this flow, which is somewhere in the upper half plane. (*b*) Prove that a stream function exists, and then find $\psi(x, y)$, using the hint that $\int dx/(a^2 + x^2) = (1/a)\tan^{-1}(x/a)$.

P4.77 Outside an inner, intense-activity circle of radius R, a tropical storm can be simulated by a polar-coordinate velocity potential $\phi(r, \theta) = U_o R\,\theta$, where U_o is the wind velocity at radius R. (*a*) Determine the velocity components outside $r = R$. (*b*) If, at $R = 25$ mi, the velocity is 100 mi/h and the pressure 99 kPa, calculate the velocity and pressure at $r = 100$ mi.

P4.78 An incompressible, irrotational, two-dimensional flow has the following stream function in polar coordinates:

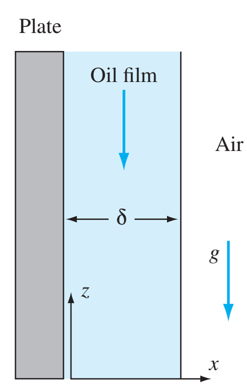
$$\psi = A\,r^n \sin(n\theta) \qquad \text{where } A \text{ and } n \text{ are constants.}$$

Find an expression for the velocity potential of this flow.

Incompressible viscous flows

***P4.79** Study the combined effect of the two viscous flows in Fig. 4.12. That is, find $u(y)$ when the upper plate moves at speed V and there is also a constant pressure gradient (dp/dx). Is superposition possible? If so, explain why. Plot representative velocity profiles for (*a*) zero, (*b*) positive, and (*c*) negative pressure gradients for the same upper-wall speed V.

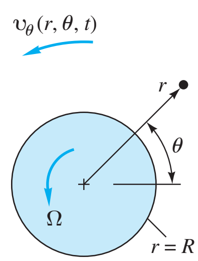
***P4.80** Oil, of density ρ and viscosity μ, drains steadily down the side of a vertical plate, as in Fig. P4.80. After a development region near the top of the plate, the oil film will become independent of z and of constant thickness δ. Assume that $w = w(x)$ only and that the atmosphere offers no shear resistance to the surface of the film. (*a*) Solve the Navier-Stokes equation for $w(x)$, and sketch its approximate shape. (*b*) Suppose that film thickness δ and the slope of the velocity profile at the wall $[\partial w/\partial x]_{wall}$ are measured with a laser-Doppler anemometer (Chap. 6). Find an expression for oil viscosity μ as a function of $(\rho, \delta, g, [\partial w/\partial x]_{wall})$.



P4.80

P4.81 Modify the analysis of Fig. 4.13 to find the velocity u_θ when the inner cylinder is fixed and the outer cylinder rotates at angular velocity Ω_0. May this solution be *added* to Eq. (4.140) to represent the flow caused when both inner and outer cylinders rotate? Explain your conclusion.

***P4.82** A solid circular cylinder of radius R rotates at angular velocity Ω in a viscous incompressible fluid that is at rest far from the cylinder, as in Fig. P4.82. Make simplifying assumptions and derive the governing differential equation and boundary conditions for the velocity field υ_θ in the fluid. Do not solve unless you are obsessed with this problem. What is the steady-state flow field for this problem?

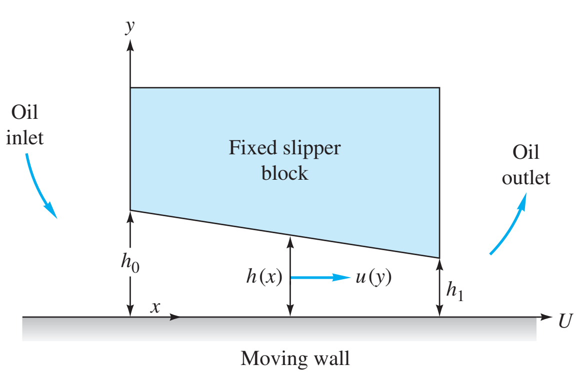


P4.82

P4.83 The flow pattern in bearing lubrication can be illustrated by Fig. P4.83, where a viscous oil (ρ, μ) is forced into the gap $h(x)$ between a fixed slipper block and a wall moving at velocity U. If the gap is thin, $h \ll L$, it can be shown that the pressure and velocity distributions are of the form $p = p(x)$, $u = u(y)$, $\upsilon = w = 0$. Neglecting gravity, reduce the Navier-Stokes equations (4.38) to a single differential equation for $u(y)$. What are the proper boundary conditions? Integrate and show that

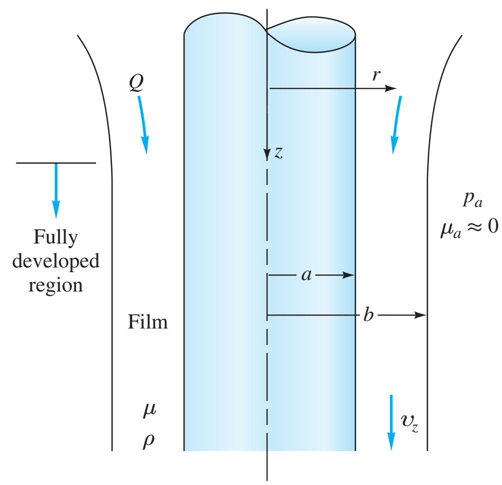
$$u = \frac{1}{2\mu}\frac{dp}{dx}(y^2 - yh) + U\left(1 - \frac{y}{h}\right)$$

where $h = h(x)$ may be an arbitrary, slowly varying gap width. (For further information on lubrication theory, see Ref. 16.)



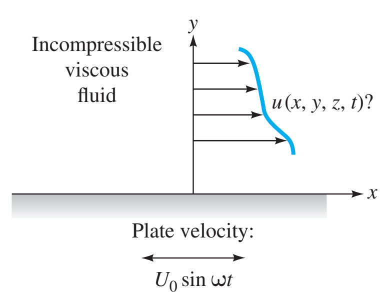
P4.83

***P4.84** Consider a viscous film of liquid draining uniformly down the side of a vertical rod of radius a, as in Fig. P4.84. At some distance down the rod the film will approach a terminal or *fully developed* draining flow of constant outer radius b, with $\upsilon_z = \upsilon_z(r)$, $\upsilon_\theta = \upsilon_r = 0$. Assume that the atmosphere offers no shear resistance to the film motion. Derive a differential equation for υ_z, state the proper boundary conditions, and solve for the film velocity distribution. How does the film radius b relate to the total film volume flow rate Q?



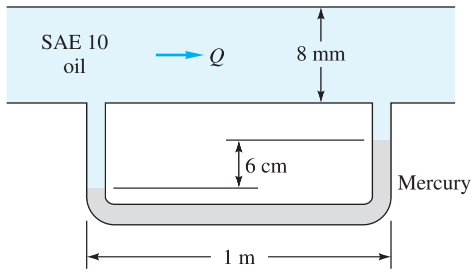
P4.84

P4.85 A flat plate of essentially infinite width and breadth oscillates sinusoidally in its own plane beneath a viscous fluid, as in Fig. P4.85. The fluid is at rest far above the plate. Making as many simplifying assumptions as you can, set up the governing differential equation and boundary conditions for finding the velocity field u in the fluid. Do not solve (if you *can* solve it immediately, you might be able to get exempted from the balance of this course with credit).



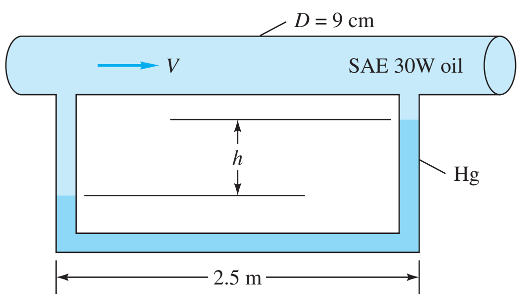
P4.85

P4.86 SAE 10 oil at 20°C flows between parallel plates 8 mm apart, as in Fig. P4.86. A mercury manometer, with wall pressure taps 1 m apart, registers a 6-cm height, as shown. Estimate the flow rate of oil for this condition.



P4.86

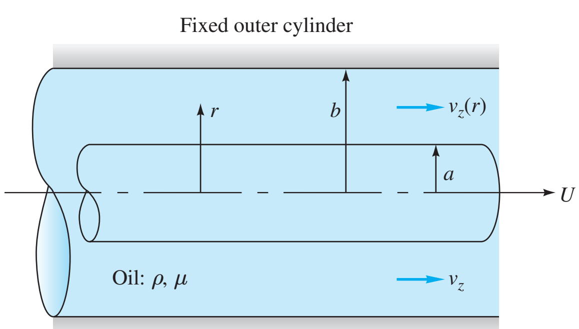
P4.87 SAE 30W oil at 20°C flows through the 9-cm-diameter pipe in Fig. P4.87 at an average velocity of 4.3 m/s.



P4.87

(*a*) Verify that the flow is laminar. (*b*) Determine the volume flow rate in m^3/h. (*c*) Calculate the expected reading h of the mercury manometer, in cm.

P4.88 The viscous oil in Fig. P4.88 is set into steady motion by a concentric inner cylinder moving axially at velocity U inside a fixed outer cylinder. Assuming constant pressure and density and a purely axial fluid motion, solve Eqs. (4.38) for the fluid velocity distribution $\upsilon_z(r)$. What are the proper boundary conditions?



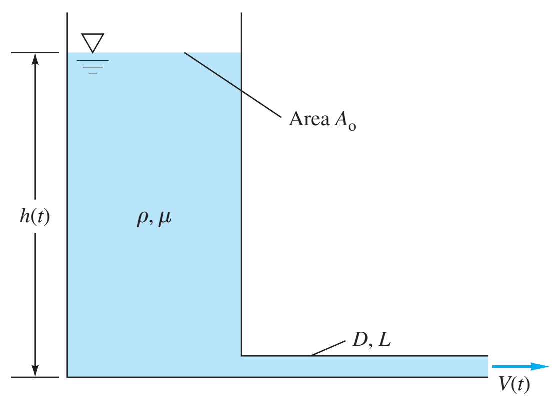
P4.88

P4.89 Oil flows steadily between two fixed plates that are 2 inches apart. When the pressure gradient is 3200 pascals per meter, the average velocity is 0.8 m/s. (*a*) What is the flow rate per meter of width? (*b*) What oil in Table A.4 fits this data? (*c*) Can we be sure that the flow is laminar?

P4.90 It is desired to pump ethanol at 20°C through 25 m of straight smooth tubing under laminar-flow conditions, $Re_d = \rho V d/\mu < 2300$. The available pressure drop is 10 kPa. (*a*) What is the maximum possible mass flow, in kg/h? (*b*) What is the appropriate diameter?

***P4.91** Analyze fully developed laminar pipe flow for a *power-law* fluid, $\tau = C(dv_z/dr)^n$, for $n \neq 1$, as in Prob. P1.46. (*a*) Derive an expression for $v_z(r)$. (*b*) For extra credit, plot the velocity profile shapes for n = 0.5, 1, and 2. [*Hint:* In Eq. (4.136), replace $\mu(dv_z/dr)$ with τ.]

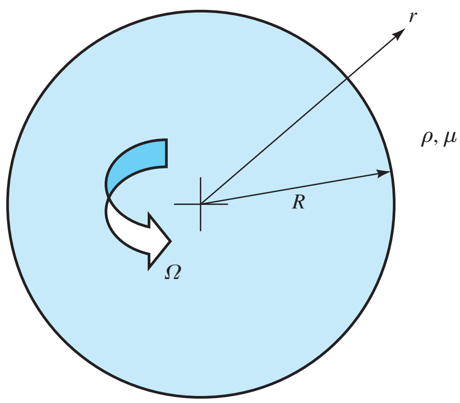
P4.92 A tank of area A_0 is draining in laminar flow through a pipe of diameter D and length L, as shown in Fig. P4.92. Neglecting the exit jet kinetic energy and assuming the pipe flow is driven by the hydrostatic pressure at its entrance, derive a formula for the tank level $h(t)$ if its initial level is h_0.



P4.92

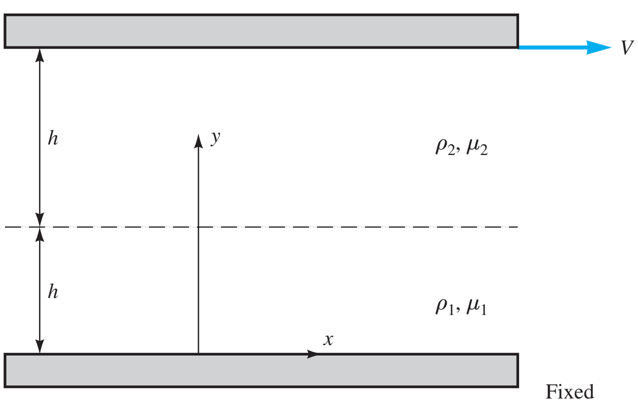
P4.93 A number of straight 25-cm-long microtubes of diameter d are bundled together into a "honeycomb" whose total cross-sectional area is 0.0006 m^2. The pressure drop from entrance to exit is 1.5 kPa. It is desired that the total volume flow rate be 5 m^3/h of water at 20°C. (*a*) What is the appropriate microtube diameter? (*b*) How many microtubes are in the bundle? (*c*) What is the Reynolds number of each microtube?

P4.94 A long, solid cylinder rotates steadily in a very viscous fluid, as in Fig. P4.94. Assuming laminar flow, solve the Navier-Stokes equation in polar coordinates to determine the resulting velocity distribution. The fluid is at rest far from the cylinder. [*Hint:* The cylinder does not induce any radial motion.]



P4.94

***P4.95** Two immiscible liquids of equal thickness h are being sheared between a fixed and a moving plate, as in Fig. P4.95. Gravity is neglected, and there is no variation with x. Find an expression for (*a*) the velocity at the interface and (*b*) the shear stress in each fluid. Assume steady laminar flow.



P4.95

P4.96 Use the data of Prob. P1.40, with the inner cylinder rotating and outer cylinder fixed, and calculate (*a*) the inner shear stress. (*b*) Determine whether this flow pattern is stable. [*Hint:* The shear stress in (r, θ) coordinates is *not* like plane flow.

Slip flows

P4.97 For Couette flow between a moving and a fixed plate, Fig. 4.12*a*, solve continuity and Navier-Stokes to find the velocity distribution when there is *slip* at both walls.

P4.98 For the pressure-gradient flow between two parallel plates of Fig. 4.12(*b*), reanalyze for the case of *slip flow* at both walls. Use the simple slip condition $u_{wall} = \ell(du/dy)_{wall}$, where ℓ is the mean free path of the fluid. (*a*) Sketch the expected velocity profile. (*b*) Find an expression for the shear stress at each wall. (*c*) Find the volume flow between the plates.

P4.99 For the pressure-gradient flow in a circular tube in Sec. 4.10, reanalyze for the case of *slip flow* at the wall. Use the simple slip condition $v_{z,wall} = \ell(dv_z/dr)_{wall,}$ where ℓ is the mean free path of the fluid. (*a*) Sketch the expected velocity profile. (*b*) Find an expression for the shear stress at the wall. (*c*) Find the volume flow through the tube.

Word Problems

W4.1 The total acceleration of a fluid particle is given by Eq. (4.2) in the Eulerian[?] system, where **V** is a known function of space and time. Explain how we might evaluate particle acceleration in the Lagrangian[?] frame, where particle position **r** is a known function of time and initial position, $\mathbf{r} = \text{fcn}(\mathbf{r}_0, t)$. Can you give an illustrative example?

W4.2 Is it true that the continuity relation, Eq. (4.6), is valid for both viscous and inviscid, newtonian and nonnewtonian, compressible and incompressible flow? If so, are there *any* limitations on this equation?

W4.3 Consider a CD (compact disc) rotating at angular velocity Ω. Does it have *vorticity* in the sense of this chapter? If so, how much vorticity?

W4.4 How much acceleration can fluids endure? Are fluids like astronauts, who feel that $5g$ is severe? Perhaps use the flow pattern of Example 4.8, at $r = R$, to make some estimates of fluid acceleration magnitudes.

W4.5 State the conditions (there are more than one) under which the analysis of temperature distribution in a flow field can be completely uncoupled, so that a separate analysis for velocity and pressure is possible. Can we do this for both laminar and turbulent flow?

W4.6 Consider liquid flow over a dam or weir. How might the boundary conditions and the flow pattern change when we compare water flow over a large prototype to SAE 30 oil flow over a tiny scale model?

W4.7 What is the difference between the stream function ψ and our method of finding the streamlines from Sec. 1.11? Or are they essentially the same?

W4.8 Under what conditions do both the stream function ψ and the velocity potential ϕ exist for a flow field? When does one exist but not the other?

W4.9 How might the remarkable three-dimensional Taylor instability of Fig. 4.14 be predicted? Discuss a general procedure for examining the stability of a given flow pattern.

W4.10 Consider an irrotational, incompressible, axisymmetric ($\partial/\partial\theta = 0$) flow in ($r$, z) coordinates. Does a stream function exist? If so, does it satisfy Laplace's equation? Are lines of constant ψ equal to the flow streamlines? Does a velocity potential exist? If so, does it satisfy Laplace's equation? Are lines of constant ϕ everywhere perpendicular to the ψ lines?

Fundamentals of Engineering Exam Problems

This chapter is not a favorite of the people who prepare the FE Exam. Probably not a single problem from this chapter will appear on the exam, but if some did, they might be like these.

FE4.1 Given the steady, incompressible velocity distribution $\mathbf{V} = 3x\mathbf{i} + Cy\mathbf{j} + 0\mathbf{k}$, where C is a constant, if conservation of mass is satisfied, the value of C should be
(*a*) 3, (*b*) 3/2, (*c*) 0, (*d*) −3/2, (*e*) −3

FE4.2 Given the steady velocity distribution $\mathbf{V} = 3x\mathbf{i} + 0\mathbf{j} + Cy\mathbf{k}$, where C is a constant, if the flow is irrotational, the value of C should be
(*a*) 3, (*b*) 3/2, (*c*) 0, (*d*) −3/2, (*e*) −3

FE4.3 Given the steady, incompressible velocity distribution $\mathbf{V} = 3x\mathbf{i} + Cy\mathbf{j} + 0\mathbf{k}$, where C is a constant, the shear stress τ_{xy} at the point (x, y, z) is given by
(*a*) 3μ, (*b*) $(3x + Cy)\mu$, (*c*) 0, (*d*) $C\mu$, (*e*) $(3 + C)\mu$

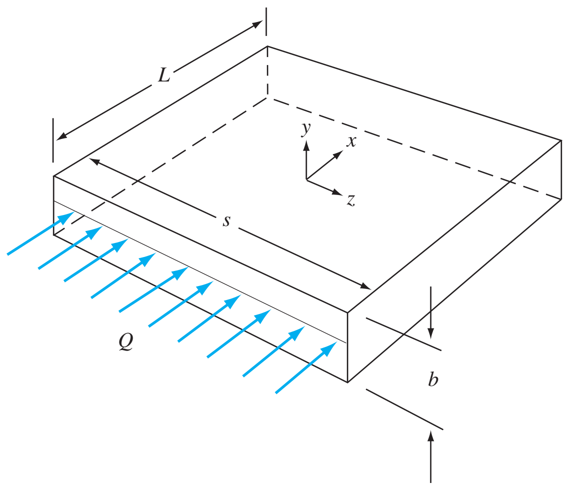
FE4.4 Given the steady, incompressible velocity distribution $u = Ax$, $v = By$, and $w = Cxy$, where (A, B, C) are constants. This flow satisfies the equation of continuity if A equals
(*a*) B, (*b*) $B + C$, (*c*) $B - C$, (*d*) $-B$, (*e*) $-(B + C)$

FE4.5 For the velocity field in Prob. FE4.4, the convective acceleration in the x direction is
(*a*) Ax^2, (*b*) A^2x, (*c*) B^2y, (*d*) By^2, (*e*) Cx^2y

FE4.6 If, for laminar flow in a smooth, straight tube, the tube diameter and length both double, while everything else remains the same, the volume flow rate will increase by a factor of
(*a*) 2, (*b*) 4, (*c*) 8, (*d*) 12, (*e*) 16

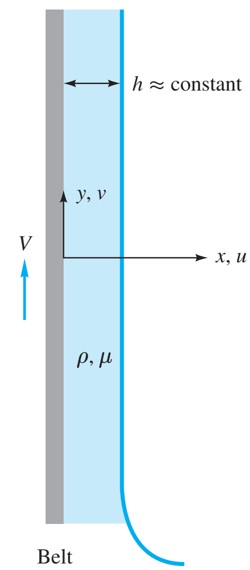
Comprehensive Problems

C4.1 In a certain medical application, water at room temperature and pressure flows through a rectangular channel of length L = 10 cm, width s = 1.0 cm, and gap thickness b = 0.30 mm as in Fig. C4.1. The volume flow rate is sinusoidal with amplitude $\hat{Q}$ = 0.50 mL/s and frequency f = 20 Hz, i.e., $Q = \hat{Q} \sin(2\pi ft)$.
(*a*) Calculate the maximum Reynolds number (Re = Vb/v) based on maximum average velocity and gap thickness. Channel flow like this remains laminar for Re less than about 2000. If Re is greater than about 2000, the flow will be turbulent. Is this flow laminar or turbulent? (*b*) In this problem, the frequency is low enough that at any given time, the flow can be solved as if it were steady at the given flow rate. (This is called a *quasi-steady assumption.*) At any arbitrary instant of time, find an expression for streamwise velocity u as a function of y, μ, dp/dx, and b, where dp/dx is the pressure gradient required to push the flow through the channel at volume flow rate Q. In addition, estimate the maximum magnitude of velocity component u. (*c*) At any instant of time, find a relationship between volume flow rate Q and pressure gradient dp/dx. Your answer should be given as an expression for Q as a function of dp/dx, s, b, and viscosity μ. (*d*) Estimate the wall shear stress, τ_w as a function of $\hat{Q}$, f, μ, b, s, and time (t). (*e*) Finally, for the numbers given in the problem statement, estimate the amplitude of the wall shear stress, $\hat{\tau}_w$, in N/m^2.



C4.1

C4.2 A belt moves upward at velocity *V*, dragging a film of viscous liquid of thickness *h*, as in Fig. C4.2. Near the belt, the film moves upward due to no slip. At its outer edge, the film moves downward due to gravity. Assuming that the only nonzero velocity is $\upsilon(x)$, with zero shear stress at the outer film edge, derive a formula for (*a*) $\upsilon(x)$, (*b*) the average velocity V_{avg} in the film, and (*c*) the velocity V_c for which there is no net flow either up or down. (*d*) Sketch $\upsilon(x)$ for case (*c*).



C4.2

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