# Chapter 5
# Dimensional Analysis and Similarity

**Motivation.** In this chapter we discuss the planning, presentation, and interpretation of experimental data. We shall try to convince you that such data are best presented in *dimensionless* form. Experiments that might result in tables of output, or even multiple volumes of tables, might be reduced to a single set of curves—or even a single curve—when suitably nondimensionalized. The technique for doing this is *dimensional analysis.* It is also effective in theoretical studies.

Chapter 3 presented large-scale control volume balances of mass, momentum, and energy, which led to global results: mass flow, force, torque, total work done, or heat transfer. Chapter 4 presented infinitesimal balances that led to the basic partial differential equations of fluid flow and some particular solutions for both inviscid and viscous (laminar) flow. These straight *analytical* techniques are limited to simple geometries and uniform boundary conditions. Only a fraction of engineering flow problems can be solved by direct analytical formulas.

Most practical fluid flow problems are too complex, both geometrically and physically, to be solved analytically. They must be tested by experiment or approximated by computational fluid dynamics (CFD) [2]. These results are typically reported as experimental or numerical data points and smoothed curves. Such data have much more generality if they are expressed in compact, economic form. This is the motivation for dimensional analysis. The technique is a mainstay of fluid mechanics and is also widely used in all engineering fields plus the physical, biological, medical, and social sciences. The present chapter shows how dimensional analysis improves the presentation of both data and theory.

## 5.1 Introduction

Basically, dimensional analysis is a method for reducing the number and complexity of experimental variables that affect a given physical phenomenon, by using a sort of compacting technique. If a phenomenon depends on $n$ dimensional variables, dimensional analysis will reduce the problem to only $k$ *dimensionless* variables, where the reduction

$n - k = 1, 2, 3$, or 4, depending on the problem complexity. Generally $n - k$ equals the number of different dimensions (sometimes called basic or primary or fundamental dimensions) that govern the problem. In fluid mechanics, the four basic dimensions are usually taken to be mass $M$, length $L$, time $T$, and temperature $\Theta$, or an $MLT\Theta$ system for short. Alternatively, one uses an $FLT\Theta$ system, with force $F$ replacing mass.

Although its purpose is to reduce variables and group them in dimensionless form, dimensional analysis has several side benefits. The first is enormous savings in time and money. Suppose one knew that the force $F$ on a particular body shape immersed in a stream of fluid depended only on the body length $L$, stream velocity $V$, fluid density $\rho$, and fluid viscosity $\mu$; that is,

$$F = f(L, V, \rho, \mu) \tag{5.1}$$

Suppose further that the geometry and flow conditions are so complicated that our integral theories (Chap. 3) and differential equations (Chap. 4) fail to yield the solution for the force. Then we must find the function $f(L, V, \rho, \mu)$ experimentally or numerically.

Generally speaking, it takes about 10 points to define a curve. To find the effect of body length in Eq. (5.1), we have to run the experiment for 10 lengths $L$. For each $L$ we need 10 values of $V$, 10 values of $\rho$, and 10 values of $\mu$, making a grand total of $10^4$, or 10,000, experiments. At \$100 per experiment—well, you see what we are getting into. However, with dimensional analysis, we can immediately reduce Eq. (5.1) to the equivalent form

$$\frac{F}{\rho V^2 L^2} = g\left(\frac{\rho V L}{\mu}\right)$$

or

$$C_F = g(\text{Re}) \tag{5.2}$$

That is, the dimensionless *force coefficient* $F/(\rho V^2 L^2)$ is a function only of the dimensionless *Reynolds number* $\rho VL/\mu$. We shall learn exactly how to make this reduction in Secs. 5.2 and 5.3. Equation (5.2) will be useful in Chap. 7.

Note that Eq. (5.2) is just an *example,* not the full story, of forces caused by fluid flows. Some fluid forces have a very weak or negligible Reynolds number dependence in wide regions (Fig. 5.3*a*). Other groups may also be important. The force coefficient may depend, in high-speed gas flow, on the *Mach number*, $\text{Ma} = V/a$, where $a$ is the speed of sound. In free-surface flows, such as ship drag, $C_F$ may depend upon *Froude number*, $\text{Fr} = V^2/(gL)$, where $g$ is the acceleration of gravity. In turbulent flow, force may depend upon the *roughness ratio*, $\epsilon/L$, where $\epsilon$ is the roughness height of the surface.

The function $g$ is different mathematically from the original function $f$, but it contains all the same information. Nothing is lost in a dimensional analysis. And think of the savings: We can establish $g$ by running the experiment for only 10 values of the single variable called the Reynolds number. We do not have to vary $L$, $V$, $\rho$, or $\mu$ separately but only the *grouping* $\rho VL/\mu$. This we do merely by varying velocity $V$ in, say, a wind tunnel or drop test or water channel, and there is no need to build 10 different bodies or find 100 different fluids with 10 densities and 10 viscosities. The cost is now about \$1000, maybe less.

A second side benefit of dimensional analysis is that it helps our thinking and planning for an experiment or theory. It suggests dimensionless ways of writing equations before we spend money on computer analysis to find solutions. It suggests variables that can be discarded; sometimes dimensional analysis will immediately reject variables, and at other times it groups them off to the side, where a few simple tests will show them to be unimportant. Finally, dimensional analysis will often give a great deal of insight into the form of the physical relationship we are trying to study.

A third benefit is that dimensional analysis provides *scaling laws* that can convert data from a cheap, small *model* to design information for an expensive, large *prototype*. We do not build a million-dollar airplane and see whether it has enough lift force. We measure the lift on a small model and use a scaling law to predict the lift on the full-scale prototype airplane. There are rules we shall explain for finding scaling laws. When the scaling law is valid, we say that a condition of *similarity* exists between the model and the prototype. In the simple case of Eq. (5.1), similarity is achieved if the Reynolds number is the same for the model and prototype because the function $g$ then requires the force coefficient to be the same also:

$$\text{If } \mathrm{Re}_m = \mathrm{Re}_p \quad \text{then} \quad C_{Fm} = C_{Fp} \tag{5.3}$$

where subscripts $m$ and $p$ mean model and prototype, respectively. From the definition of force coefficient, this means that

$$\frac{F_p}{F_m} = \frac{\rho_p}{\rho_m}\left(\frac{V_p}{V_m}\right)^2\left(\frac{L_p}{L_m}\right)^2 \tag{5.4}$$

for data taken where $\rho_p V_p L_p/\mu_p = \rho_m V_m L_m/\mu_m$. Equation (5.4) is a scaling law: If you measure the model force at the model Reynolds number, the prototype force at the same Reynolds number equals the model force times the density ratio times the velocity ratio squared times the length ratio squared. We shall give more examples later.

Do you understand these introductory explanations? Be careful; learning dimensional analysis is like learning to play tennis: There are levels of the game. We can establish some ground rules and do some fairly good work in this brief chapter, but dimensional analysis in the broad view has many subtleties and nuances that only time, practice, and maturity enable you to master. Although dimensional analysis has a firm physical and mathematical foundation, considerable art and skill are needed to use it effectively.

**EXAMPLE 5.1**

A copepod is a water crustacean approximately 1 mm in diameter. We want to know the drag force on the copepod when it moves slowly in fresh water. A scale model 100 times larger is made and tested in glycerin at $V = 30$ cm/s. The measured drag on the model is 1.3 N. For similar conditions, what are the velocity and drag of the actual copepod in water? Assume that Eq. (5.2) applies and the temperature is 20°C.

**Solution**

- *Property values:* From Table A.3, the densities and viscosities at 20°C are

  Water (prototype): $\mu_p = 0.001$ kg/(m-s) $\rho_p = 998$ kg/m$^3$

  Glycerin (model): $\mu_m = 1.5$ kg/(m-s) $\rho_m = 1263$ kg/m$^3$

- *Assumptions:* Equation (5.2) is appropriate and *similarity* is achieved; that is, the model and prototype have the same Reynolds number and, therefore, the same force coefficient.

- *Approach:* The length scales are $L_m = 100$ mm and $L_p = 1$ mm. Calculate the Reynolds number and force coefficient of the model and set them equal to prototype values:

$$\text{Re}_m = \frac{\rho_m V_m L_m}{\mu_m} = \frac{(1263\ \text{kg/m}^3)(0.3\ \text{m/s})(0.1\ \text{m})}{1.5\ \text{kg/(m-s)}} = 25.3 = \text{Re}_p = \frac{(998\ \text{kg/m}^3)V_p(0.001\ \text{m})}{0.001\ \text{kg/(m-s)}}$$

$$\text{Solve for } V_p = 0.0253\ \text{m/s} = 2.53\ \text{cm/s} \qquad \textit{Ans.}$$

In like manner, using the prototype velocity just found, equate the force coefficients:

$$C_{Fm} = \frac{F_m}{\rho_m V_m^2 L_m^2} = \frac{1.3\ \text{N}}{(1263\ \text{kg/m}^3)(0.3\ \text{m/s})^2(0.1\ \text{m})^2} = 1.14$$

$$= C_{Fp} = \frac{F_p}{(998\ \text{kg/m}^3)(0.0253\ \text{m/s})^2(0.001\ \text{m})^2}$$

$$\text{Solve for } F_p = 7.3\text{E-7 N} \qquad \textit{Ans.}$$

- *Comments:* Assuming we modeled the Reynolds number correctly, the model test is a very good idea, as it would obviously be difficult to measure such a tiny copepod drag force.

Historically, the first person to write extensively about units and dimensional reasoning in physical relations was Euler in 1765. Euler's ideas were far ahead of his time, as were those of Joseph Fourier, whose 1822 book *Analytical Theory of Heat* outlined what is now called the *principle of dimensional homogeneity* and even developed some similarity rules for heat flow. There were no further significant advances until Lord Rayleigh's book in 1877, *Theory of Sound,* which proposed a "method of dimensions" and gave several examples of dimensional analysis. The final breakthrough that established the method as we know it today is generally credited to E. Buckingham in 1914 [1], whose paper outlined what is now called the *Buckingham Pi Theorem* for describing dimensionless parameters (see Sec. 5.3). However, it is now known that a Frenchman, A. Vaschy, in 1892 and a Russian, D. Riabouchinsky, in 1911 had independently published papers reporting results equivalent to the pi theorem. Following Buckingham's paper, P. W. Bridgman published a classic book in 1922 [3], outlining the general theory of dimensional analysis.

Dimensional analysis is so valuable and subtle, with both skill and art involved, that it has spawned a wide variety of textbooks and treatises. The writer is aware of more than 30 books on the subject, of which his engineering favorites are listed here [3–10]. Dimensional analysis is not confined to fluid mechanics, or even to engineering. Specialized books have been published on the application of dimensional analysis to metrology [11], astrophysics [12], economics [13], chemistry [14], hydrology [15], medications [16], clinical medicine [17], chemical processing pilot plants [18], social sciences [19], biomedical sciences [20], pharmacy [21], fractal geometry [22], and even the growth of plants [23]. Clearly this is a subject well worth learning for many career paths.

## 5.2 The Principle of Dimensional Homogeneity

In making the remarkable jump from the five-variable Eq. (5.1) to the two-variable Eq. (5.2), we were exploiting a rule that is almost a self-evident axiom in physics. This rule, the *principle of dimensional homogeneity* (PDH), can be stated as follows:

If an equation truly expresses a proper relationship between variables in a physical process, it will be *dimensionally homogeneous;* that is, each of its additive terms will have the same dimensions.

All the equations that are derived from the theory of mechanics are of this form. For example, consider the relation that expresses the displacement of a falling body:

$$S = S_0 + V_0 t + \tfrac{1}{2} g t^2 \tag{5.5}$$

Each term in this equation is a displacement, or length, and has dimensions $\{L\}$. The equation is dimensionally homogeneous. Note also that any consistent set of units can be used to calculate a result.

Consider Bernoulli's equation for incompressible flow:

$$\frac{p}{\rho} + \frac{1}{2} V^2 + gz = \text{const} \tag{5.6}$$

Each term, including the constant, has dimensions of velocity squared, or $\{L^2 T^{-2}\}$. The equation is dimensionally homogeneous and gives proper results for any consistent set of units.

Students count on dimensional homogeneity and use it to check themselves when they cannot quite remember an equation during an exam. For example, which is it:

$$S = \tfrac{1}{2} g t^2? \qquad \text{or} \qquad S = \tfrac{1}{2} g^2 t? \tag{5.7}$$

By checking the dimensions, we reject the second form and back up our faulty memory. We are exploiting the principle of dimensional homogeneity, and this chapter simply exploits it further.

## Variables and Constants

Equations (5.5) and (5.6) also illustrate some other factors that often enter into a dimensional analysis:

*Dimensional variables* are the quantities that actually vary during a given case and would be plotted against each other to show the data. In Eq. (5.5), they are $S$ and $t$; in Eq. (5.6) they are $p$, $V$, and $z$. All have dimensions, and all can be nondimensionalized as a dimensional analysis technique.

*Dimensional constants* may vary from case to case but are held constant during a given run. In Eq. (5.5) they are $S_0$, $V_0$, and $g$, and in Eq. (5.6) they are $\rho$, $g$, and $C$. They all have dimensions and conceivably could be nondimensionalized, but they are normally used to help nondimensionalize the variables in the problem.

*Pure constants* have no dimensions and never did. They arise from mathematical manipulations. In both Eqs. (5.5) and (5.6) they are $\frac{1}{2}$ and the exponent 2, both of which came from an integration: $\int t\,dt = \frac{1}{2}t^2$, $\int V\,dV = \frac{1}{2}V^2$. Other common dimensionless constants are $\pi$ and $e$. Also, the argument of any mathematical function, such as ln, exp, cos, or $J_0$, is dimensionless.

*Angles* and *revolutions* are dimensionless. The preferred unit for an angle is the radian, which makes it clear that an angle is a ratio. In like manner, a revolution is $2\pi$ radians.

*Counting numbers* are dimensionless. For example, if we triple the energy $E$ to $3E$, the coefficient 3 is dimensionless.

Note that integration and differentiation of an equation may change the dimensions but not the homogeneity of the equation. For example, integrate or differentiate Eq. (5.5):

$$\int S\,dt = S_0 t + \tfrac{1}{2}V_0 t^2 + \tfrac{1}{6}g t^3 \tag{5.8a}$$

$$\frac{dS}{dt} = V_0 + gt \tag{5.8b}$$

In the integrated form (5.8*a*) every term has dimensions of $\{LT\}$, while in the derivative form (5.8*b*) every term is a velocity $\{LT^{-1}\}$.

Finally, some physical variables are naturally dimensionless by virtue of their definition as ratios of dimensional quantities. Some examples are strain (change in length per unit length), Poisson's ratio (ratio of transverse strain to longitudinal strain), and specific gravity (ratio of density to standard water density).

The motive behind dimensional analysis is that any dimensionally homogeneous equation can be written in an entirely equivalent nondimensional form that is more compact. Usually there are multiple methods of presenting one's dimensionless data or theory. Let us illustrate these concepts more thoroughly by using the falling-body relation (5.5) as an example.

## Ambiguity: The Choice of Variables and Scaling Parameters[1]

Equation (5.5) is familiar and simple, yet it illustrates most of the concepts of dimensional analysis. It contains five terms ($S$, $S_0$, $V_0$, $t$, $g$), which we may divide, in our thinking, into variables and parameters. The *variables* are the things we wish to plot, the basic output of the experiment or theory: in this case, $S$ versus $t$. The *parameters* are those quantities whose effect on the variables we wish to know: in this case $S_0$, $V_0$, and $g$. Almost any engineering study can be subdivided in this manner.

To nondimensionalize our results, we need to know how many dimensions are contained among our variables and parameters: in this case, only two, length $\{L\}$ and time $\{T\}$. Check each term to verify this:

$$\{S\} = \{S_0\} = \{L\} \qquad \{t\} = \{T\} \qquad \{V_0\} = \{LT^{-1}\} \qquad \{g\} = \{LT^{-2}\}$$

Among our parameters, we therefore select two to be *scaling parameters* (also called *repeating variables*), used to define dimensionless variables. What remains will be the "basic" parameter(s) whose effect we wish to show in our plot. These choices will not affect the content of our data, only the form of their presentation. Clearly there is ambiguity in these choices, something that often vexes the beginning experimenter. But the ambiguity is deliberate. Its purpose is to show a particular effect, and the choice is yours to make.

For the falling-body problem, we select any two of the three parameters to be scaling parameters. Thus, we have three options. Let us discuss and display them in turn.

[1] I am indebted to Prof. Jacques Lewalle of Syracuse University for suggesting, outlining, and clarifying this entire discussion.

***Option 1:*** Scaling parameters $S_0$ and $V_0$: the effect of gravity $g$.

First use the scaling parameters ($S_0$, $V_0$) to define dimensionless (*) displacement and time. There is only one suitable definition for each:[2]

$$S^* = \frac{S}{S_0} \qquad t^* = \frac{V_0 t}{S_0} \tag{5.9}$$

Substitute these variables into Eq. (5.5) and clean everything up until each term is dimensionless. The result is our first option:

$$S^* = 1 + t^* + \frac{1}{2}\alpha t^{*2} \qquad \alpha = \frac{gS_0}{V_0^2} \tag{5.10}$$

This result is shown plotted in Fig. 5.1*a*. There is a single dimensionless parameter $\alpha$, which shows here the effect of gravity. It cannot show the direct effects of $S_0$ and $V_0$, since these two are hidden in the ordinate and abscissa. We see that gravity increases the parabolic rate of fall for $t^* > 0$, but not the initial slope at $t^* = 0$. We would learn the same from falling-body data, and the plot, within experimental accuracy, would look like Fig. 5.1*a*.

***Option 2:*** Scaling parameters $V_0$ and $g$: the effect of initial displacement $S_0$.

Now use the new scaling parameters ($V_0$, $g$) to define dimensionless (**) displacement and time. Again there is only one suitable definition:

$$S^{**} = \frac{Sg}{V_0^2} \qquad t^{**} = t\,\frac{g}{V_0} \tag{5.11}$$

Substitute these variables into Eq. (5.5) and clean everything up again. The result is our second option:

$$S^{**} = \alpha + t^{**} + \frac{1}{2}t^{**2} \qquad \alpha = \frac{gS_0}{V_0^2} \tag{5.12}$$

This result is plotted in Fig. 5.1*b*. The same single parameter $\alpha$ again appears and here shows the effect of initial *displacement,* which merely moves the curves upward without changing their shape.

***Option 3:*** Scaling parameters $S_0$ and $g$: the effect of initial speed $V_0$.

Finally use the scaling parameters ($S_0$, $g$) to define dimensionless (***) displacement and time. Again there is only one suitable definition:

$$S^{***} = \frac{S}{S_0} \qquad t^{***} = t\left(\frac{g}{S_0}\right)^{1/2} \tag{5.13}$$

Substitute these variables into Eq. (5.5) and clean everything up as usual. The result is our third and final option:

$$S^{***} = 1 + \beta t^{***} + \frac{1}{2}t^{***2} \qquad \beta = \frac{1}{\sqrt{\alpha}} = \frac{V_0}{\sqrt{gS_0}} \tag{5.14}$$

[2]Make them *proportional* to $S$ and $t$. Do not define dimensionless terms upside down: $S_0/S$ or $S_0/(V_0 t)$. The plots will look funny, users of your data will be confused, and your supervisor will be angry. It is not a good idea.

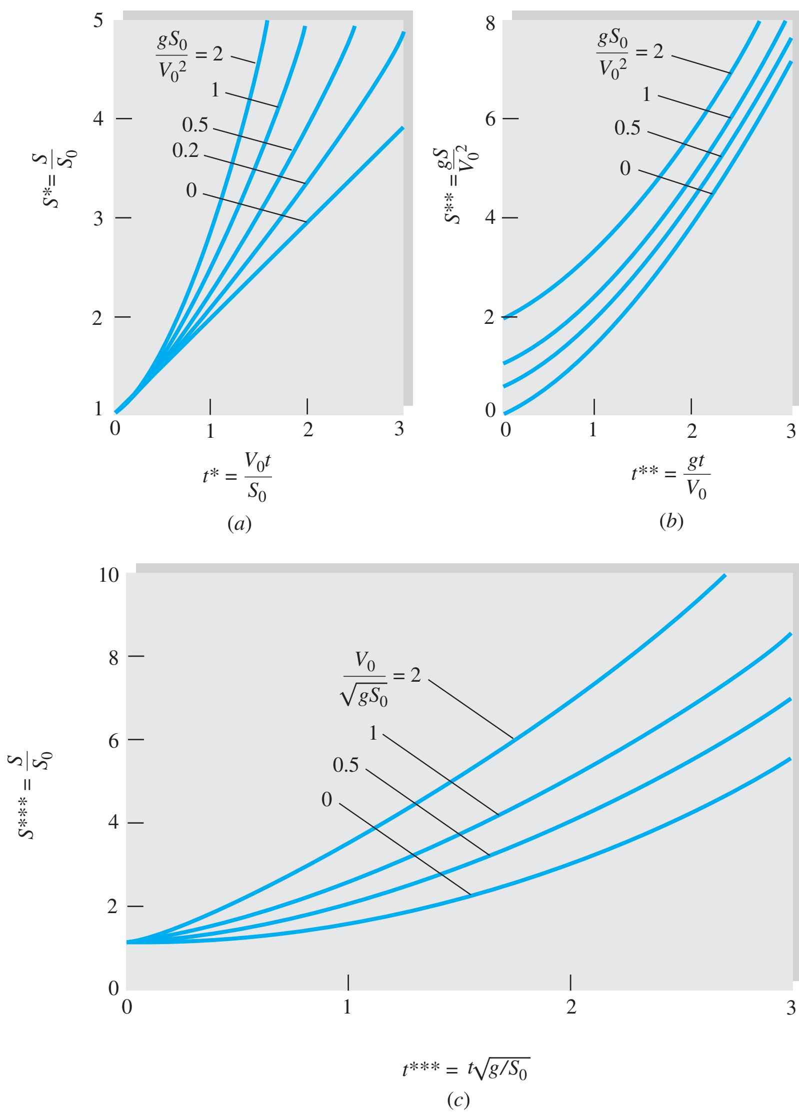



**Fig. 5.1** Three entirely equivalent dimensionless presentations of the falling-body problem, Eq. (5.5): the effect of (*a*) gravity, (*b*) initial displacement, and (*c*) initial velocity. All plots contain the same information.

This final presentation is shown in Fig. 5.1*c*. Once again the parameter $\alpha$ appears, but we have redefined it upside down, $\beta = 1/\sqrt{\alpha}$, so that our display parameter $V_0$ is in the numerator and is linear. This is our free choice and simply improves the display. Figure 5.1*c* shows that initial *velocity* increases the falling displacement.

Note that, in all three options, the same parameter $\alpha$ appears but has a different meaning: dimensionless gravity, initial displacement, and initial velocity. The graphs, which contain exactly the same information, change their appearance to reflect these differences.

Whereas the original problem, Eq. (5.5), involved five quantities, the dimensionless presentations involve only three, having the form

$$S' = \text{fcn}(t', \alpha) \qquad \alpha = \frac{gS_0}{V_0^2} \tag{5.15}$$

The reduction 5 − 3 = 2 should equal the number of fundamental dimensions involved in the problem $\{L, T\}$. This idea led to the pi theorem (Sec. 5.3).

**Selection of Scaling (Repeating) Variables**

The selection of scaling variables is left to the user, but there are some guidelines. In Eq. (5.2), it is now clear that the scaling variables were $\rho$, $V$, and $L$, since they appear in both force coefficient and Reynolds number. We could then interpret data from Eq. (5.2) as the variation of dimensionless *force* versus dimensionless *viscosity,* since each appears in only one dimensionless group. Similarly, in Eq. (5.5) the scaling variables were selected from ($S_0$, $V_0$, $g$), not ($S$, $t$), because we wished to plot $S$ versus $t$ in the final result.

The following are some guidelines for selecting scaling variables:

1. They must *not* form a dimensionless group among themselves, but adding one more variable *will* form a dimensionless quantity. For example, test powers of $\rho$, $V$, and $L$:

$$\rho^a V^b L^c = (ML^{-3})^a (L/T)^b (L)^c = M^0 L^0 T^0 \text{ only if } a = 0, b = 0, c = 0$$

   In this case, we can see why this is so: Only $\rho$ contains the dimension $\{M\}$, and only $V$ contains the dimension $\{T\}$, so no cancellation is possible. If, now, we add $\mu$ to the scaling group, we will obtain the Reynolds number. If we add $F$ to the group, we form the force coefficient.
2. Do not select output variables for your scaling parameters. In Eq. (5.1), certainly do not select $F$, which you wish to isolate for your plot. Nor was $\mu$ selected, for we wished to plot force versus viscosity.
3. If convenient, select *popular,* not obscure, scaling variables because they will appear in all of your dimensionless groups. Select density, not surface tension. Select body length, not surface roughness. Select stream velocity, not speed of sound.

The examples that follow will make this clear. Problem assignments might give hints.

Suppose we wish to study drag force versus *velocity*. Then we would not use $V$ as a scaling parameter in Eq. (5.1). We would use ($\rho$, $\mu$, $L$) instead, and the final dimensionless function would become

$$C_F' = \frac{\rho F}{\mu^2} = f(\text{Re}) \qquad \text{Re} = \frac{\rho V L}{\mu} \tag{5.16}$$

In plotting these data, we would not be able to discern the effect of $\rho$ or $\mu$, since they appear in both dimensionless groups. The grouping $C_F'$ again would mean dimensionless force, and Re is now interpreted as either dimensionless velocity or size.[3] The plot would be quite different compared to Eq. (5.2), although it contains exactly the same information. The development of parameters such as $C_F'$ and Re from the initial variables is the subject of the pi theorem (Sec. 5.3).

**Some Peculiar Engineering Equations**

The foundation of the dimensional analysis method rests on two assumptions: (1) The proposed physical relation is dimensionally homogeneous, and (2) all the relevant variables have been included in the proposed relation.

If a relevant variable is missing, dimensional analysis will fail, giving either algebraic difficulties or, worse, yielding a dimensionless formulation that does not resolve

[3]We were lucky to achieve a size effect because in this case $L$, a scaling parameter, did not appear in the drag coefficient.

the process. A typical case is Manning's open-channel formula, discussed in Example 1.4 and Chap. 10.

$$V = \frac{1.49}{n} R^{2/3} S^{1/2} \tag{1}$$

Since $V$ is velocity, $R$ is a radius, and $n$ and $S$ are dimensionless, the formula is not dimensionally homogeneous. This should be a warning that (1) the formula changes if the *units* of $V$ and $R$ change and (2) if valid, it represents a very special case. Equation (1) in Example 1.4 predates the dimensional analysis technique and is valid only for water in rough channels at moderate velocities and large radii in BG units.

Such dimensionally inhomogeneous formulas abound in the hydraulics literature. Another example is the Hazen-Williams formula [24] for volume flow of water through a straight smooth pipe:

$$Q = 61.9 D^{2.63} \left(\frac{dp}{dx}\right)^{0.54} \tag{5.17}$$

where $D$ is diameter and $dp/dx$ is the pressure gradient. Some of these formulas arise because numbers have been inserted for fluid properties and other physical data into perfectly legitimate homogeneous formulas. We shall not give the units of Eq. (5.17) to avoid encouraging its use.

On the other hand, some formulas are "constructs" that cannot be made dimensionally homogeneous. The "variables" they relate cannot be analyzed by the dimensional analysis technique. Most of these formulas are raw empiricisms convenient to a small group of specialists. Here are three examples:

$$B = \frac{25{,}000}{100 - R} \tag{5.18}$$

$$S = \frac{140}{130 + \text{API}} \tag{5.19}$$

$$0.0147 D_E - \frac{3.74}{D_E} = 0.26 t_R - \frac{172}{t_R} \tag{5.20}$$

Equation (5.18) relates the Brinell hardness $B$ of a metal to its Rockwell hardness $R$. Equation (5.19) relates the specific gravity $S$ of an oil to its density in degrees API. Equation (5.20) relates the viscosity of a liquid in $D_E$, or degrees Engler, to its viscosity $t_R$ in Saybolt seconds. Such formulas have a certain usefulness when communicated between fellow specialists, but we cannot handle them here. Variables like Brinell hardness and Saybolt viscosity are not suited to an $MLT\Theta$ dimensional system.

## 5.3 The Pi Theorem

There are several methods of reducing a number of dimensional variables into a smaller number of dimensionless groups. The first scheme given here was proposed in 1914 by Buckingham [1] and is now called the *Buckingham Pi Theorem*. The name *pi* comes from the mathematical notation $\Pi$, meaning a product of variables. The dimensionless groups found from the theorem are power products denoted by $\Pi_1$, $\Pi_2$, $\Pi_3$, etc. The method allows the pi groups to be found in sequential order without resorting to free exponents.

The first part of the pi theorem explains what reduction in variables to expect:

> If a physical process satisfies the PDH and involves $n$ dimensional variables, it can be reduced to a relation between $k$ dimensionless variables or $\Pi$s. The reduction $j = n - k$ equals the maximum number of variables that do not form a pi among themselves and is always less than or equal to the number of dimensions describing the variables.

Take the specific case of force on an immersed body: Eq. (5.1) contains five variables $F$, $L$, $U$, $\rho$, and $\mu$ described by three dimensions $\{MLT\}$. Thus $n = 5$ and $j \leq 3$. Therefore it is a good guess that we can reduce the problem to $k$ pi groups, with $k = n - j \geq 5 - 3 = 2$. And this is exactly what we obtained: two dimensionless variables $\Pi_1 = C_F$ and $\Pi_2 = \text{Re}$. On rare occasions it may take more pi groups than this minimum (see Example 5.5).

The second part of the theorem shows how to find the pi groups one at a time:

> Find the reduction $j$, then select $j$ scaling variables that do not form a pi among themselves.[4] Each desired pi group will be a power product of these $j$ variables plus one additional variable, which is assigned any convenient nonzero exponent. Each pi group thus found is independent.

To be specific, suppose the process involves five variables:

$$\upsilon_1 = f(\upsilon_2, \upsilon_3, \upsilon_4, \upsilon_5)$$

Suppose there are three dimensions $\{MLT\}$ and we search around and find that indeed $j = 3$. Then $k = 5 - 3 = 2$ and we expect, from the theorem, two and only two pi groups. Pick out three convenient variables that do *not* form a pi, and suppose these turn out to be $\upsilon_2$, $\upsilon_3$, and $\upsilon_4$. Then the two pi groups are formed by power products of these three plus one additional variable, either $\upsilon_1$ or $\upsilon_5$:

$$\Pi_1 = (\upsilon_2)^a(\upsilon_3)^b(\upsilon_4)^c\upsilon_1 = M^0L^0T^0 \quad \Pi_2 = (\upsilon_2)^a(\upsilon_3)^b(\upsilon_4)^c\upsilon_5 = M^0L^0T^0$$

Here we have arbitrarily chosen $\upsilon_1$ and $\upsilon_5$, the added variables, to have unit exponents. Equating exponents of the various dimensions is guaranteed by the theorem to give unique values of $a$, $b$, and $c$ for each pi. And they are independent because only $\Pi_1$ contains $\upsilon_1$ and only $\Pi_2$ contains $\upsilon_5$. It is a very neat system once you get used to the procedure. We shall illustrate it with several examples.

Typically, six steps are involved:

1. List and count the $n$ variables involved in the problem. If any important variables are missing, dimensional analysis will fail.
2. List the dimensions of each variable according to $\{MLT\Theta\}$ or $\{FLT\Theta\}$. A list is given in Table 5.1.
3. Find $j$. Initially guess $j$ equal to the number of different dimensions present, and look for $j$ variables that do not form a pi product. If no luck, reduce $j$ by 1 and look again. With practice, you will find $j$ rapidly.
4. Select $j$ scaling parameters that do not form a pi product. Make sure they please you and have some generality if possible, because they will then appear

[4] Make a clever choice here because all pi groups will contain these $j$ variables in various groupings.

**Table 5.1** Dimensions of Fluid-Mechanics Properties

| Quantity | Symbol | Dimensions $MLT\Theta$ | Dimensions $FLT\Theta$ |
|---|---|---|---|
| Length | $L$ | $L$ | $L$ |
| Area | $A$ | $L^2$ | $L^2$ |
| Volume | $\mathcal{V}$ | $L^3$ | $L^3$ |
| Velocity | $V$ | $LT^{-1}$ | $LT^{-1}$ |
| Acceleration | $dV/dt$ | $LT^{-2}$ | $LT^{-2}$ |
| Speed of sound | $a$ | $LT^{-1}$ | $LT^{-1}$ |
| Volume flow | $Q$ | $L^3T^{-1}$ | $L^3T^{-1}$ |
| Mass flow | $\dot{m}$ | $MT^{-1}$ | $FTL^{-1}$ |
| Pressure, stress | $p, \sigma, \tau$ | $ML^{-1}T^{-2}$ | $FL^{-2}$ |
| Strain rate | $\dot{\varepsilon}$ | $T^{-1}$ | $T^{-1}$ |
| Angle | $\theta$ | None | None |
| Angular velocity | $\omega, \Omega$ | $T^{-1}$ | $T^{-1}$ |
| Viscosity | $\mu$ | $ML^{-1}T^{-1}$ | $FTL^{-2}$ |
| Kinematic viscosity | $\nu$ | $L^2T^{-1}$ | $L^2T^{-1}$ |
| Surface tension | $\Upsilon$ | $MT^{-2}$ | $FL^{-1}$ |
| Force | $F$ | $MLT^{-2}$ | $F$ |
| Moment, torque | $M$ | $ML^2T^{-2}$ | $FL$ |
| Power | $P$ | $ML^2T^{-3}$ | $FLT^{-1}$ |
| Work, energy | $W, E$ | $ML^2T^{-2}$ | $FL$ |
| Density | $\rho$ | $ML^{-3}$ | $FT^2L^{-4}$ |
| Temperature | $T$ | $\Theta$ | $\Theta$ |
| Specific heat | $c_p, c_v$ | $L^2T^{-2}\Theta^{-1}$ | $L^2T^{-2}\Theta^{-1}$ |
| Specific weight | $\gamma$ | $ML^{-2}T^{-2}$ | $FL^{-3}$ |
| Thermal conductivity | $k$ | $MLT^{-3}\Theta^{-1}$ | $FT^{-1}\Theta^{-1}$ |
| Thermal expansion coefficient | $\beta$ | $\Theta^{-1}$ | $\Theta^{-1}$ |

in every one of your pi groups. Pick density or velocity or length. Do not pick surface tension, for example, or you will form six different independent Weber-number parameters and thoroughly annoy your colleagues.

5. Add one additional variable to your $j$ repeating variables, and form a power product. Algebraically find the exponents that make the product dimensionless. Try to arrange for your output or *dependent* variables (force, pressure drop, torque, power) to appear in the numerator, and your plots will look better. Do this sequentially, adding one new variable each time, and you will find all $n - j = k$ desired pi products.
6. Write the final dimensionless function, and check the terms to make sure all pi groups are dimensionless.

**EXAMPLE 5.2**

Repeat the development of Eq. (5.2) from Eq. (5.1), using the pi theorem.

**Solution**

**Step 1** Write the function and count variables:

$$F = f(L, U, \rho, \mu) \quad \text{there are five variables } (n = 5)$$

**Step 2** List dimensions of each variable. From Table 5.1

| $F$ | $L$ | $U$ | $\rho$ | $\mu$ |
|---|---|---|---|---|
| $\{MLT^{-2}\}$ | $\{L\}$ | $\{LT^{-1}\}$ | $\{ML^{-3}\}$ | $\{ML^{-1}T^{-1}\}$ |

**Step 3** Find $j$. No variable contains the dimension $\Theta$, and so $j$ is less than or equal to 3 ($MLT$). We inspect the list and see that $L$, $U$, and $\rho$ cannot form a pi group because only $\rho$ contains mass and only $U$ contains time. Therefore $j$ does equal 3, and $n - j = 5 - 3 = 2 = k$. The pi theorem guarantees for this problem that there will be exactly two independent dimensionless groups.

**Step 4** Select repeating $j$ variables. The group $L$, $U$, $\rho$ we found in step 3 will do fine.

**Step 5** Combine $L$, $U$, $\rho$ with one additional variable, in sequence, to find the two pi products.

First add force to find $\Pi_1$. You may select *any* exponent on this additional term as you please, to place it in the numerator or denominator to any power. Since $F$ is the output, or dependent, variable, we select it to appear to the first power in the numerator:

$$\Pi_1 = L^a U^b \rho^c F = (L)^a (LT^{-1})^b (ML^{-3})^c (MLT^{-2}) = M^0 L^0 T^0$$

Equate exponents:

Length: $a + b - 3c + 1 = 0$

Mass: $c + 1 = 0$

Time: $-b \quad -2 = 0$

We can solve explicitly for

$$a = -2 \qquad b = -2 \qquad c = -1$$

Therefore $$\Pi_1 = L^{-2} U^{-2} \rho^{-1} F = \frac{F}{\rho U^2 L^2} = C_F$$ *Ans.*

This is exactly the right pi group as in Eq. (5.2). By varying the exponent on $F$, we could have found other equivalent groups such as $UL\rho^{1/2}/F^{1/2}$.

Finally, add viscosity to $L$, $U$, and $\rho$ to find $\Pi_2$. Select any power you like for viscosity. By hindsight and custom, we select the power $-1$ to place it in the denominator:

$$\Pi_2 = L^a U^b \rho^c \mu^{-1} = L^a (LT^{-1})^b (ML^{-3})^c (ML^{-1}T^{-1})^{-1} = M^0 L^0 T^0$$

Equate exponents:

Length: $a + b - 3c + 1 = 0$

Mass: $c - 1 = 0$

Time: $-b \quad + 1 = 0$

from which we find

$$a = b = c = 1$$

Therefore $$\Pi_2 = L^1 U^1 \rho^1 \mu^{-1} = \frac{\rho U L}{\mu} = \text{Re}$$ *Ans.*

**Step 6** We know we are finished; this is the second and last pi group. The theorem guarantees that the functional relationship must be of the equivalent form

$$\frac{F}{\rho U^2 L^2} = g\left(\frac{\rho U L}{\mu}\right) \qquad \textit{Ans.}$$

which is exactly Eq. (5.2).

## EXAMPLE 5.3

The power input $P$ to a centrifugal pump is a function of the volume flow $Q$, impeller diameter $D$, rotational rate $\Omega$, and the density $\rho$ and viscosity $\mu$ of the fluid:

$$P = f(Q, D, \Omega, \rho, \mu)$$

Rewrite this as a dimensionless relationship. *Hint:* Use $\Omega$, $\rho$, and $D$ as repeating variables. We will revisit this problem in Chap. 11.

### Solution

**Step 1** Count the variables. There are six (don't forget the one on the left, $P$).

**Step 2** List the dimensions of each variable from Table 5.1. Use the $\{FLT\Theta\}$ system:

| $P$ | $Q$ | $D$ | $\Omega$ | $\rho$ | $\mu$ |
|---|---|---|---|---|---|
| $\{FLT^{-1}\}$ | $\{L^3T^{-1}\}$ | $\{L\}$ | $\{T^{-1}\}$ | $\{FT^2L^{-4}\}$ | $\{FTL^{-2}\}$ |

**Step 3** Find $j$. Lucky us, we were told to use $(\Omega, \rho, D)$ as repeating variables, so surely $j = 3$, the number of dimensions $(FLT)$? Check that these three do *not* form a pi group:

$$\Omega^a \rho^b D^c = (T^{-1})^a (FT^2L^{-4})^b (L)^c = F^0L^0T^0 \text{ only if } \qquad a = 0, b = 0, c = 0$$

Yes, $j = 3$. This was not as obvious as the scaling group $(L, U, \rho)$ in Example 5.2, but it is true. We now know, from the theorem, that adding one more variable will indeed form a pi group.

**Step 4a** Combine $(\Omega, \rho, D)$ with power $P$ to find the first pi group:

$$\Pi_1 = \Omega^a \rho^b D^c P = (T^{-1})^a (FT^2L^{-4})^b (L)^c (FLT^{-1}) = F^0L^0T^0$$

Equate exponents:

$$\begin{aligned}
&\text{Force:} & b \quad\quad + 1 &= 0 \\
&\text{Length:} & -4b + c + 1 &= 0 \\
&\text{Time:} & -a + 2b \quad\quad - 1 &= 0
\end{aligned}$$

Solve algebraically to obtain $a = -3$, $b = -1$, and $c = -5$. This first pi group, the output dimensionless variable, is called the *power coefficient* of a pump, $C_P$:

$$\Pi_1 = \Omega^{-3}\rho^{-1}D^{-5}P = \frac{P}{\rho \Omega^3 D^5} = C_P$$

**Step 4b** Combine $(\Omega, \rho, D)$ with flow rate $Q$ to find the second pi group:

$$\Pi_2 = \Omega^a \rho^b D^c Q = (T^{-1})^a (FT^2L^{-4})^b (L)^c (L^3T^{-1}) = F^0L^0T^0$$

After equating exponents, we now find $a = -1$, $b = 0$, and $c = -3$. This second pi group is called the *flow coefficient* of a pump, $C_Q$:

$$\Pi_2 = \Omega^{-1}\rho^0 D^{-3} Q = \frac{Q}{\Omega D^3} = C_Q$$

**Step 4c** Combine $(\Omega, \rho, D)$ with viscosity $\mu$ to find the third and last pi group:

$$\Pi_3 = \Omega^a \rho^b D^c \mu = (T^{-1})^a (FT^2L^{-4})^b (L)^c (FTL^{-2}) = F^0L^0T^0$$

This time, $a = -1$, $b = -1$, and $c = -2$; or $\Pi_3 = \mu/(\rho\Omega D^2)$, a sort of Reynolds number.

**Step 5** The original relation between six variables is now reduced to three dimensionless groups:

$$\frac{P}{\rho\Omega^3 D^5} = f\left(\frac{Q}{\Omega D^3}, \frac{\mu}{\rho\Omega D^2}\right) \qquad \textit{Ans.}$$

*Comment:* These three are the classical coefficients used to correlate pump power in Chap. 11.

## EXAMPLE 5.4

At low velocities (laminar flow), the volume flow $Q$ through a small-bore tube is a function only of the tube radius $R$, the fluid viscosity $\mu$, and the pressure drop per unit tube length $dp/dx$. Using the pi theorem, find an appropriate dimensionless relationship.

### Solution

Write the given relation and count variables:

$$Q = f\left(R, \mu, \frac{dp}{dx}\right) \quad \text{four variables } (n = 4)$$

Make a list of the dimensions of these variables from Table 5.1 using the $\{MLT\}$ system:

| $Q$ | $R$ | $\mu$ | $dp/dx$ |
|---|---|---|---|
| $\{L^3T^{-1}\}$ | $\{L\}$ | $\{ML^{-1}T^{-1}\}$ | $\{ML^{-2}T^{-2}\}$ |

There are three primary dimensions $(M, L, T)$, hence $j \le 3$. By trial and error we determine that $R$, $\mu$, and $dp/dx$ cannot be combined into a pi group. Then $j = 3$, and $n - j = 4 - 3 = 1$. There is only *one* pi group, which we find by combining $Q$ in a power product with the other three:

$$\Pi_1 = R^a \mu^b \left(\frac{dp}{dx}\right)^c Q^1 = (L)^a (ML^{-1}T^{-1})^b (ML^{-2}T^{-2})^c (L^3T^{-1})$$
$$= M^0L^0T^0$$

Equate exponents:

Mass: $b + c = 0$

Length: $a - b - 2c + 3 = 0$

Time: $-b - 2c - 1 = 0$

Solving simultaneously, we obtain $a = -4$, $b = 1$, and $c = -1$. Then

$$\Pi_1 = R^{-4}\mu^1\left(\frac{dp}{dx}\right)^{-1} Q$$

or
$$\Pi_1 = \frac{Q\mu}{R^4(dp/dx)} = \text{const} \qquad \textit{Ans.}$$

Since there is only one pi group, it must equal a dimensionless constant. This is as far as dimensional analysis can take us. The laminar flow theory of Sec. 4.10 shows that the value of the constant is $-\frac{\pi}{8}$. This result is also useful in Chap. 6.

## EXAMPLE 5.5

Assume that the tip deflection $\delta$ of a cantilever beam is a function of the tip load $P$, beam length $L$, area moment of inertia $I$, and material modulus of elasticity $E$; that is, $\delta = f(P, L, I, E)$. Rewrite this function in dimensionless form, and comment on its complexity and the peculiar value of $j$.

### Solution

List the variables and their dimensions:

| $\delta$ | $P$ | $L$ | $I$ | $E$ |
|---|---|---|---|---|
| $\{L\}$ | $\{MLT^{-2}\}$ | $\{L\}$ | $\{L^4\}$ | $\{ML^{-1}T^{-2}\}$ |

There are five variables ($n = 5$) and three primary dimensions ($M$, $L$, $T$), hence $j \le 3$. But try as we may, we *cannot* find any combination of three variables that does not form a pi group. This is because $\{M\}$ and $\{T\}$ occur only in $P$ and $E$ and only in the same form, $\{MT^{-2}\}$. Thus we have encountered a special case of $j = 2$, which is less than the number of dimensions ($M$, $L$, $T$). To gain more insight into this peculiarity, you should rework the problem, using the ($F$, $L$, $T$) system of dimensions. You will find that only $\{F\}$ and $\{L\}$ occur in these variables, hence $j = 2$.

With $j = 2$, we select $L$ and $E$ as two variables that cannot form a pi group and then add other variables to form the three desired pis:

$$\Pi_1 = L^aE^bI^1 = (L)^a(ML^{-1}T^{-2})^b(L^4) = M^0L^0T^0$$

from which, after equating exponents, we find that $a = -4$, $b = 0$, or $\Pi_1 = I/L^4$. Then

$$\Pi_2 = L^aE^bP^1 = (L)^a(ML^{-1}T^{-2})^b(MLT^{-2}) = M^0L^0T^0$$

from which we find $a = -2$, $b = -1$, or $\Pi_2 = P/(EL^2)$, and

$$\Pi_3 = L^aE^b\delta^1 = (L)^a(ML^{-1}T^{-2})^b(L) = M^0L^0T^0$$

from which $a = -1$, $b = 0$, or $\Pi_3 = \delta/L$. The proper dimensionless function is $\Pi_3 = f(\Pi_2, \Pi_1)$, or

$$\frac{\delta}{L} = f\left(\frac{P}{EL^2}, \frac{I}{L^4}\right) \qquad \textit{Ans. } (1)$$

This is a complex three-variable function, but dimensional analysis alone can take us no further.

*Comments:* We can "improve" Eq. (1) by taking advantage of some physical reasoning, as Langhaar points out [4, p. 91]. For small elastic deflections, $\delta$ is proportional to load $P$ and inversely proportional to moment of inertia $I$. Since $P$ and $I$ occur separately in Eq. (1), this means that $\Pi_3$ must be proportional to $\Pi_2$ and inversely proportional to $\Pi_1$. Thus, for these conditions,

$$\frac{\delta}{L} = (\text{const})\,\frac{P}{EL^2}\,\frac{L^4}{I}$$

or

$$\delta = (\text{const})\,\frac{PL^3}{EI} \qquad (2)$$

This could not be predicted by a pure dimensional analysis. Strength-of-materials theory predicts that the value of the constant is $\frac{1}{3}$.

## An Alternate Step-by-Step Method by Ipsen (1960)[5]

The pi theorem method, just explained and illustrated, is often called the *repeating variable method* of dimensional analysis. Select the repeating variables, add one more, and you get a pi group. The writer likes it. This method is straightforward and systematically reveals all the desired pi groups. However, there are drawbacks: (1) All pi groups contain the same repeating variables and might lack variety or effectiveness, and (2) one must (sometimes laboriously) check that the selected repeating variables do *not* form a pi group among themselves (see Prob. P5.21).

Ipsen [5] suggests an entirely different procedure, a step-by-step method that obtains all of the pi groups at once, without any counting or checking. One simply successively eliminates each dimension in the desired function by division or multiplication. Let us illustrate with the same classical drag function proposed in Eq. (5.1). Underneath the variables, write out the dimensions of each quantity.

$$\begin{array}{ccccc} F & = \text{fcn}(L, & V, & \rho, & \mu) \\ \{MLT^{-2}\} & \{L\} & \{LT^{-1}\} & \{ML^{-3}\} & \{ML^{-1}T^{-1}\} \end{array} \qquad (5.1)$$

There are three dimensions, $\{MLT\}$. Eliminate them successively by division or multiplication by a variable. Start with mass $\{M\}$. Pick a variable that contains mass and divide it into all the other variables with mass dimensions. We select $\rho$, divide, and rewrite the function (5.1):

$$\begin{array}{ccccc} \dfrac{F}{\rho} & = \text{fcn}\Big(L, & V, & \not{\rho} & \dfrac{\mu}{\rho}\Big) \\ \{L^4T^{-2}\} & \{L\} & \{LT^{-1}\} & & \{L^2T^{-1}\} \end{array} \qquad (5.1a)$$

[5]This method may be omitted without loss of continuity.

We did not divide into $L$ or $V$, which do not contain $\{M\}$. Equation (5.1$a$) at first looks strange, but it contains five distinct variables and the same information as Eq. (5.1).

We see that $\rho$ is no longer important. Thus *discard* $\rho$, and now there are only four variables. Next, eliminate time $\{T\}$ by dividing the time-containing variables by suitable powers of, say, $V$. The result is

$$\underset{\{L^2\}}{\frac{F}{\rho V^2}} = \text{fcn}\left(\underset{\{L\}}{L}, \quad \cancel{V}, \quad \underset{\{L\}}{\frac{\mu}{\rho V}}\right) \tag{5.1b}$$

Now we see that $V$ is no longer relevant. Finally, eliminate $\{L\}$ through division by, say, appropriate powers of $L$ itself:

$$\underset{\{1\}}{\frac{F}{\rho V^2 L^2}} = \text{fcn}\left(\cancel{L}, \quad \underset{\{1\}}{\frac{\mu}{\rho V L}}\right) \tag{5.1c}$$

Now $L$ by itself is no longer relevant, and so discard it also. The result is equivalent to Eq. (5.2):

$$\frac{F}{\rho V^2 L^2} = \text{fcn}\left(\frac{\mu}{\rho V L}\right) \tag{5.2}$$

In Ipsen's step-by-step method, we find the force coefficient is a function solely of the Reynolds number. We did no counting and did not find $j$. We just successively eliminated each primary dimension by division with the appropriate variables.

Recall Example 5.5, where we discovered, awkwardly, that the number of repeating variables was *less* than the number of primary dimensions. Ipsen's method avoids this preliminary check. Recall the beam-deflection problem proposed in Example 5.5 and the various dimensions:

$$\underset{\{L\}}{\delta} = f(\underset{\{MLT^{-2}\}}{P}, \quad \underset{\{L\}}{L}, \quad \underset{\{L^4\}}{I}, \quad \underset{\{ML^{-1}T^{-2}\}}{E})$$

For the first step, let us eliminate $\{M\}$ by dividing by $E$. We only have to divide into $P$:

$$\underset{\{L\}}{\delta} = f\left(\underset{\{L^2\}}{\frac{P}{E}}, \quad \underset{\{L\}}{L}, \quad \underset{\{L^4\}}{I}, \quad \cancel{E}\right)$$

We see that we may discard $E$ as no longer relevant, and the dimension $\{T\}$ has vanished along with $\{M\}$. We need only eliminate $\{L\}$ by dividing by, say, powers of $L$ itself:

$$\underset{\{1\}}{\frac{\delta}{L}} = \text{fcn}\left(\underset{\{1\}}{\frac{P}{EL^2}}, \quad \cancel{L}, \quad \underset{\{1\}}{\frac{I}{L^4}}\right)$$

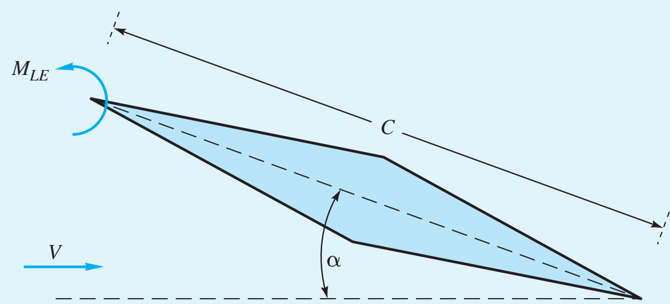
Discard $L$ itself as now irrelevant, and we obtain *Answer* (1) to Example 5.5:

$$\frac{\delta}{L} = \text{fcn}\left(\frac{P}{EL^2}, \frac{I}{L^4}\right)$$

Ipsen's approach is again successful. The fact that $\{M\}$ and $\{T\}$ vanished in the same division is proof that there are only *two* repeating variables this time, not the three that would be inferred by the presence of $\{M\}$, $\{L\}$, and $\{T\}$.

## EXAMPLE 5.6

The leading-edge aerodynamic moment $M_{LE}$ on a supersonic airfoil is a function of its chord length $C$, angle of attack $\alpha$, and several air parameters: approach velocity $V$, density $\rho$, speed of sound $a$, and specific heat ratio $k$ (Fig. E5.6). There is a very weak effect of air viscosity, which is neglected here.




**E5.6**

Use Ipsen's method to rewrite this function in dimensionless form.

### Solution

Write out the given function and list the variables' dimensions $\{MLT\}$ underneath:

$$\begin{array}{ccccccc} M_{LE} & = \text{fcn}(C, & \alpha, & V, & \rho, & a, & k) \\ \{ML^2/T^2\} & \{\text{L}\} & \{1\} & \{L/T\} & \{M/L^3\} & \{L/T\} & \{1\} \end{array}$$

Two of them, $\alpha$ and $k$, are already dimensionless. Leave them alone; they will be pi groups in the final function. You can eliminate any dimension. We choose mass $\{M\}$ and divide by $\rho$:

$$\begin{array}{ccccccc} \dfrac{M_{LE}}{\rho} & = \text{fcn}(C, & \alpha, & V, & \cancel{\rho}, & a, & k) \\ \{L^5/T^2\} & \{L\} & \{1\} & \{L/T\} & & \{L/T\} & \{1\} \end{array}$$

Recall Ipsen's rules: Only divide into variables containing mass, in this case only $M_{LE}$, and then discard the divisor, $\rho$. Now eliminate time $\{T\}$ by dividing by appropriate powers of $a$:

$$\begin{array}{cccccc} \dfrac{M_{LE}}{\rho a^2} & = \text{fcn}\Big(C, & \alpha, & \dfrac{V}{a}, & \cancel{a}, & k\Big) \\ \{L^3\} & \{\text{L}\} & \{1\} & \{1\} & & \{1\} \end{array}$$

Finally, eliminate $\{L\}$ on the left side by dividing by $C^3$:

$$\frac{M_{LE}}{\rho a^2 C^3} = \text{fcn}\left(\cancel{C}, \quad \alpha, \quad \frac{V}{a}, \quad k\right)$$

$$\{1\} \qquad\qquad\qquad \{1\} \; \{1\} \; \{1\}$$

We end up with four pi groups and recognize $V/a$ as the Mach number, Ma. In aerodynamics, the dimensionless moment is often called the *moment coefficient*, $C_M$. Thus our final result could be written in the compact form

$$C_M = \text{fcn}(\alpha, \text{Ma}, k) \qquad \textit{Ans.}$$

*Comments:* Our analysis is fine, but experiment and theory and physical reasoning all indicate that $M_{LE}$ varies more strongly with $V$ than with $a$. Thus aerodynamicists commonly define the moment coefficient as $C_M = M_{LE}/(\rho V^2 C^3)$ or something similar. We will study the analysis of supersonic forces and moments in Chap. 9.

## 5.4 Nondimensionalization of the Basic Equations

We could use the pi theorem method of the previous section to analyze problem after problem after problem, finding the dimensionless parameters that govern in each case. Textbooks on dimensional analysis [for example, 5] do this. An alternative and very powerful technique is to attack the basic equations of flow from Chap. 4. Even though these equations cannot be solved in general, they will reveal basic dimensionless parameters, such as the Reynolds number, in their proper form and proper position, giving clues to when they are negligible. The boundary conditions must also be nondimensionalized.

Let us briefly apply this technique to the incompressible flow continuity and momentum equations with constant viscosity:

Continuity:
$$\nabla \cdot \mathbf{V} = 0 \tag{5.21a}$$

Navier-Stokes:
$$\rho \frac{d\mathbf{V}}{dt} = \rho \mathbf{g} - \boldsymbol{\nabla} p + \mu \nabla^2 \mathbf{V} \tag{5.21b}$$

Typical boundary conditions for these two equations are (Sect. 4.6)

Fixed solid surface: $\mathbf{V} = 0$

Inlet or outlet: Known $\mathbf{V}$, $p$ (5.22)

Free surface, $z = \eta$: $w = \dfrac{d\eta}{dt}$ $\quad p = p_a - \Upsilon(R_x^{-1} + R_y^{-1})$

We omit the energy equation (4.75) and assign its dimensionless form in the problems (Prob. P5.43).

Equations (5.21) and (5.22) contain the three basic dimensions $M$, $L$, and $T$. All variables $p$, $\mathbf{V}$, $x$, $y$, $z$, and $t$ can be nondimensionalized by using density and two reference constants that might be characteristic of the particular fluid flow:

$$\text{Reference velocity} = U \qquad \text{Reference length} = L$$

For example, $U$ may be the inlet or upstream velocity and $L$ the diameter of a body immersed in the stream.

Now define all relevant dimensionless variables, denoting them by an asterisk:

$$\mathbf{V}^* = \frac{\mathbf{V}}{U} \qquad \nabla^* = L\nabla$$

$$x^* = \frac{x}{L} \quad y^* = \frac{y}{L} \quad z^* = \frac{z}{L} \quad R^* = \frac{R}{L} \tag{5.23}$$

$$t^* = \frac{tU}{L} \quad p^* = \frac{p + \rho g z}{\rho U^2}$$

All these are fairly obvious except for $p^*$, where we have introduced the piezometric pressure, assuming that $z$ is up. This is a hindsight idea suggested by Bernoulli's equation (3.54).

Since $\rho$, $U$, and $L$ are all constants, the derivatives in Eqs. (5.21) can all be handled in dimensionless form with dimensional coefficients. For example,

$$\frac{\partial u}{\partial x} = \frac{\partial(Uu^*)}{\partial(Lx^*)} = \frac{U}{L}\frac{\partial u^*}{\partial x^*}$$

Substitute the variables from Eqs. (5.23) into Eqs. (5.21) and (5.22) and divide through by the leading dimensional coefficient, in the same way as we handled Eq. (5.12). Here are the resulting dimensionless equations of motion:

Continuity: $$\nabla^* \cdot V^* = 0 \tag{5.24a}$$

Momentum: $$\frac{d\mathbf{V}^*}{dt^*} = -\nabla^* p^* + \frac{\mu}{\rho U L}\nabla^{*2}(\mathbf{V}^*) \tag{5.24b}$$

The dimensionless boundary conditions are:

Fixed solid surface: $\mathbf{V}^* = 0$

Inlet or outlet: Known $\mathbf{V}^*$, $p^*$

Free surface, $z^* = \eta^*$: $$w^* = \frac{d\eta^*}{dt^*}$$

$$p^* = \frac{p_a}{\rho U^2} + \frac{gL}{U^2}z^* - \frac{\Upsilon}{\rho U^2 L}(R_x^{*-1} + R_y^{*-1}) \tag{5.25}$$

These equations reveal a total of four dimensionless parameters, one in the Navier-Stokes equation and three in the free-surface-pressure boundary condition.

## Dimensionless Parameters

In the continuity equation there are no parameters. The Navier-Stokes equation contains one, generally accepted as the most important parameter in fluid mechanics:

$$\text{Reynolds number Re} = \frac{\rho U L}{\mu}$$

It is named after Osborne Reynolds (1842–1912), a British engineer who first proposed it in 1883 (Ref. 4 of Chap. 6). The Reynolds number is always important, with or without a free surface, and can be neglected only in flow regions away from high-velocity gradients—for example, away from solid surfaces, jets, or wakes.

The no-slip and inlet-exit boundary conditions contain no parameters. The free-surface-pressure condition contains three:

$$\text{Euler number (pressure coefficient) Eu} = \frac{p_a}{\rho U^2}$$

This is named after Leonhard Euler (1707–1783) and is rarely important unless the pressure drops low enough to cause vapor formation (cavitation) in a liquid. The Euler number is often written in terms of pressure differences: $\text{Eu} = \Delta p/(\rho U^2)$. If $\Delta p$ involves vapor pressure $p_v$, it is called the *cavitation number* $\text{Ca} = (p_a - p_v)/(\rho U^2)$. Cavitation problems are surprisingly common in many water flows.

The second free-surface parameter is much more important:

$$\text{Froude number Fr} = \frac{U^2}{gL}$$

It is named after William Froude (1810–1879), a British naval architect who, with his son Robert, developed the ship-model towing-tank concept and proposed similarity rules for free-surface flows (ship resistance, surface waves, open channels). The Froude number is the dominant effect in free-surface flows. It can also be important in *stratified flows*, where a strong density difference exists without a free surface. For example, see Ref. [42]. Chapter 10 investigates Froude number effects in detail.

The final free-surface parameter is

$$\text{Weber number We} = \frac{\rho U^2 L}{\Upsilon}$$

It is named after Moritz Weber (1871–1951) of the Polytechnic Institute of Berlin, who developed the laws of similitude in their modern form. It was Weber who named Re and Fr after Reynolds and Froude. The Weber number is important only if it is of order unity or less, which typically occurs when the surface curvature is comparable in size to the liquid depth, such as in droplets, capillary flows, ripple waves, and very small hydraulic models. If We is large, its effect may be neglected.

If there is no free surface, Fr, Eu, and We drop out entirely, except for the possibility of cavitation of a liquid at very small Eu. Thus, in low-speed viscous flows with no free surface, the Reynolds number is the only important dimensionless parameter.

## Compressibility Parameters

In high-speed flow of a gas there are significant changes in pressure, density, and temperature that must be related by an equation of state such as the perfect-gas law, Eq. (1.10). These thermodynamic changes introduce two additional dimensionless parameters mentioned briefly in earlier chapters:

$$\text{Mach number Ma} = \frac{U}{a} \qquad \text{Specific-heat ratio } k = \frac{c_p}{c_v}$$

The Mach number is named after Ernst Mach (1838–1916), an Austrian physicist. The effect of $k$ is only slight to moderate, but Ma exerts a strong effect on compressible flow properties if it is greater than about 0.3. These effects are studied in Chap. 9.

## Oscillating Flows

If the flow pattern is oscillating, a seventh parameter enters through the inlet boundary condition. For example, suppose that the inlet stream is of the form

$$u = U \cos \omega t$$

Nondimensionalization of this relation results in

$$\frac{u}{U} = u^* = \cos\left(\frac{\omega L}{U} t^*\right)$$

The argument of the cosine contains the new parameter

$$\text{Strouhal number St} = \frac{\omega L}{U}$$

The dimensionless forces and moments, friction, and heat transfer, and so on of such an oscillating flow would be a function of both Reynolds and Strouhal numbers. This parameter is named after V. Strouhal, a German physicist who experimented in 1878 with wires singing in the wind.

Some flows that you might guess to be perfectly steady actually have an oscillatory pattern that is dependent on the Reynolds number. An example is the periodic vortex shedding behind a blunt body immersed in a steady stream of velocity $U$. Figure 5.2*a* shows an array of alternating vortices shed from a circular cylinder immersed in a steady crossflow. This regular, periodic shedding is called a *Kármán vortex street,* after T. von Kármán, who explained it theoretically in 1912. The shedding occurs in the range $10^2 < \text{Re} < 10^7$, with an average Strouhal number $\omega d/(2\pi U) \approx 0.21$. Figure 5.2*b* shows measured shedding frequencies.

Resonance can occur if a vortex shedding frequency is near a body's structural vibration frequency. Electric transmission wires sing in the wind, undersea mooring lines gallop at certain current speeds, and slender structures flutter at critical wind or vehicle speeds. A striking example is the disastrous failure of the Tacoma Narrows suspension bridge in 1940, when wind-excited vortex shedding caused resonance with the natural torsional oscillations of the bridge. The problem was magnified by the bridge deck nonlinear stiffness, which occurred when the hangers went slack during the oscillation.

## Other Dimensionless Parameters

We have discussed seven important parameters in fluid mechanics, and there are others. Four additional parameters arise from nondimensionalization of the energy equation (4.75) and its boundary conditions. These four (Prandtl number, Eckert number, Grashof number, and wall temperature ratio) are listed in Table 5.2 just in case you fail to solve Prob. P5.43. Another important and perhaps surprising parameter is the

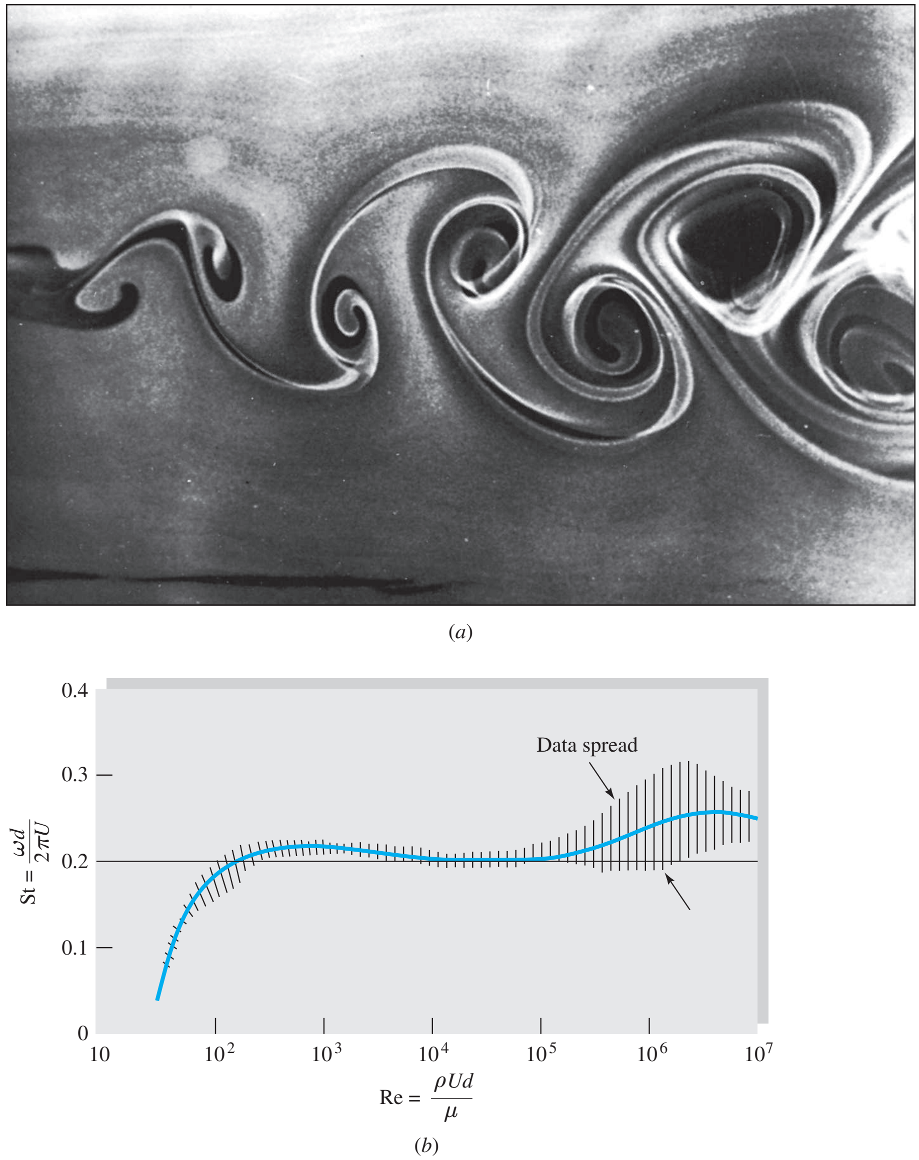



**Fig. 5.2** Vortex shedding from a circular cylinder: (*a*) vortex street behind a circular cylinder *(Courtesy of U.S. Navy)*; (*b*) experimental shedding frequencies *(data from Refs. 25 and 26)*.

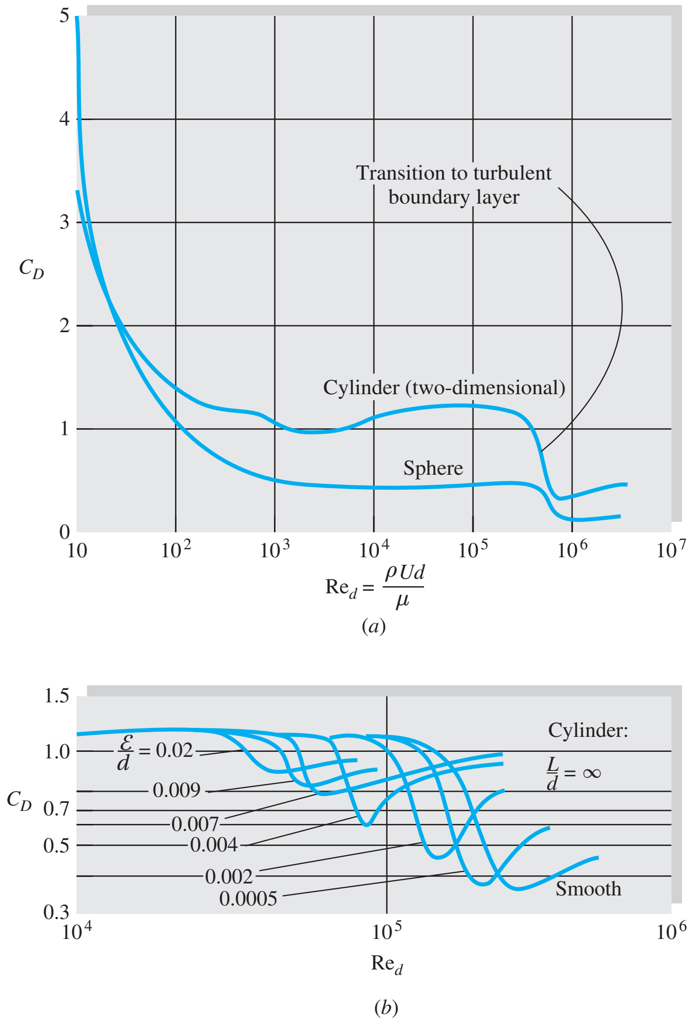
wall roughness ratio $\varepsilon/L$ (in Table 5.2).[6] Slight changes in surface roughness have a striking effect in the turbulent flow or high-Reynolds-number range, as we shall see in Chap. 6 and in Fig. 5.3.

This book is primarily concerned with Reynolds-, Mach-, and Froude-number effects, which dominate most flows. Note that we discovered these parameters (except $\varepsilon/L$) simply by nondimensionalizing the basic equations without actually solving them.

[6]Roughness is easy to overlook because it is a slight geometric effect that does not appear in the equations of motion. It is a boundary condition that one might forget.

**Table 5.2** Dimensionless Groups in Fluid Mechanics

| Parameter | Definition | Qualitative ratio of effects | Importance |
|---|---|---|---|
| Reynolds number | $\text{Re} = \frac{\rho U L}{\mu}$ | $\frac{\text{Inertia}}{\text{Viscosity}}$ | Almost always |
| Mach number | $\text{Ma} = \frac{U}{a}$ | $\frac{\text{Flow speed}}{\text{Sound speed}}$ | Compressible flow |
| Froude number | $\text{Fr} = \frac{U^2}{gL}$ | $\frac{\text{Inertia}}{\text{Gravity}}$ | Free-surface flow |
| Weber number | $\text{We} = \frac{\rho U^2 L}{\Upsilon}$ | $\frac{\text{Inertia}}{\text{Surface tension}}$ | Free-surface flow |
| Rossby number | $\text{Ro} = \frac{U}{\Omega_{\text{earth}} L}$ | $\frac{\text{Flow velocity}}{\text{Coriolis effect}}$ | Geophysical flows |
| Cavitation number (Euler number) | $\text{Ca} = \frac{p - p_v}{\frac{1}{2}\rho U^2}$ | $\frac{\text{Pressure}}{\text{Inertia}}$ | Cavitation |
| Prandtl number | $\text{Pr} = \frac{\mu c_p}{k}$ | $\frac{\text{Dissipation}}{\text{Conduction}}$ | Heat convection |
| Eckert number | $\text{Ec} = \frac{U^2}{c_p T_0}$ | $\frac{\text{Kinetic energy}}{\text{Enthalpy}}$ | Dissipation |
| Specific-heat ratio | $k = \frac{c_p}{c_v}$ | $\frac{\text{Enthalpy}}{\text{Internal energy}}$ | Compressible flow |
| Strouhal number | $\text{St} = \frac{\omega L}{U}$ | $\frac{\text{Oscillation}}{\text{Mean speed}}$ | Oscillating flow |
| Roughness ratio | $\frac{\varepsilon}{L}$ | $\frac{\text{Wall roughness}}{\text{Body length}}$ | Turbulent, rough walls |
| Grashof number | $\text{Gr} = \frac{\beta \Delta T g L^3 \rho^2}{\mu^2}$ | $\frac{\text{Buoyancy}}{\text{Viscosity}}$ | Natural convection |
| Rayleigh number | $\text{Ra} = \frac{\beta \Delta T g L^3 \rho^2 c_p}{\mu k}$ | $\frac{\text{Buoyancy}}{\text{Viscosity}}$ | Natural convection |
| Temperature ratio | $\frac{T_w}{T_0}$ | $\frac{\text{Wall temperature}}{\text{Stream temperature}}$ | Heat transfer |
| Pressure coefficient | $C_p = \frac{p - p_\infty}{\frac{1}{2}\rho U^2}$ | $\frac{\text{Static pressure}}{\text{Dynamic pressure}}$ | Aerodynamics, hydrodynamics |
| Lift coefficient | $C_L = \frac{L}{\frac{1}{2}\rho U^2 A}$ | $\frac{\text{Lift force}}{\text{Dynamic force}}$ | Aerodynamics, hydrodynamics |
| Drag coefficient | $C_D = \frac{D}{\frac{1}{2}\rho U^2 A}$ | $\frac{\text{Drag force}}{\text{Dynamic force}}$ | Aerodynamics, hydrodynamics |
| Friction factor | $f = \frac{h_f}{(V^2/2g)(L/d)}$ | $\frac{\text{Friction head loss}}{\text{Velocity head}}$ | Pipe flow |
| Skin friction coefficient | $c_f = \frac{\tau_{\text{wall}}}{\rho V^2/2}$ | $\frac{\text{Wall shear stress}}{\text{Dynamic pressure}}$ | Boundary layer flow |




Cylinder length effect

($10^4 < \mathrm{Re} < 10^5$)

| $L/d$ | $C_D$ |
|---|---|
| $\infty$ | 1.20 |
| 40 | 0.98 |
| 20 | 0.91 |
| 10 | 0.82 |
| 5 | 0.74 |
| 3 | 0.72 |
| 2 | 0.68 |
| 1 | 0.64 |

**Fig. 5.3** The proof of practical dimensional analysis: drag coefficients of a cylinder and sphere: (*a*) drag coefficient of a smooth cylinder and sphere (data from many sources); (*b*) increased roughness causes earlier transition to a turbulent boundary layer.

If the reader is not satiated with the 19 parameters given in Table 5.2, Ref. 29 contains a list of over 1200 dimensionless parameters in use in engineering and science.

## A Successful Application

Dimensional analysis is fun, but does it work? Yes, if all important variables are included in the proposed function, the dimensionless function found by dimensional analysis will collapse all the data onto a single curve or set of curves.

An example of the success of dimensional analysis is given in Fig. 5.3 for the measured drag on smooth cylinders and spheres. The flow is normal to the axis of the cylinder, which is extremely long, $L/d \rightarrow \infty$. The data are from many sources, for both liquids and gases, and include bodies from several meters in diameter down to fine wires and balls less than 1 mm in size. Both curves in Fig. 5.3*a* are entirely experimental; the analysis of immersed body drag is one of the weakest areas of modern fluid mechanics theory. Except for digital computer calculations, there is little theory for cylinder and sphere drag except *creeping flow,* $\mathrm{Re} < 1$.

The concept of a fluid-caused *drag force* on bodies is covered extensively in Chap. 7. Drag is the fluid force parallel to the oncoming stream—see Fig. 7.10 for details.

The Reynolds number of both bodies is based on diameter, hence the notation $\mathrm{Re}_d$. But the drag coefficients are defined differently:

$$C_D = \begin{cases} \dfrac{\text{drag}}{\frac{1}{2}\rho U^2 L d} & \text{cylinder} \\ \dfrac{\text{drag}}{\frac{1}{2}\rho U^2 \frac{1}{4}\pi d^2} & \text{sphere} \end{cases} \tag{5.26}$$

They both have a factor $\frac{1}{2}$ because the term $\frac{1}{2}\rho U^2$ occurs in Bernoulli's equation, and both are based on the projected area—that is, the area one sees when looking toward the body from upstream. The usual definition of $C_D$ is thus

$$C_D = \frac{\text{drag}}{\frac{1}{2}\rho U^2(\text{projected area})} \tag{5.27}$$

However, one should carefully check the definitions of $C_D$, Re, and the like before using data in the literature. Airfoils, for example, use the planform area.

Figure 5.3*a* is for long, smooth cylinders. If wall roughness and cylinder length are included as variables, we obtain from dimensional analysis a complex three-parameter function:

$$C_D = f\left(\mathrm{Re}_d, \frac{\varepsilon}{d}, \frac{L}{d}\right) \tag{5.28}$$

To describe this function completely would require 1000 or more experiments or CFD results. Therefore it is customary to explore the length and roughness effects separately to establish trends.

The table with Fig. 5.3*a* shows the length effect with zero wall roughness. As length decreases, the drag decreases by up to 50 percent. Physically, the pressure is "relieved" at the ends as the flow is allowed to skirt around the tips instead of deflecting over and under the body.

Figure 5.3*b* shows the effect of wall roughness for an infinitely long cylinder. The sharp drop in drag occurs at lower $\mathrm{Re}_d$ as roughness causes an earlier transition to a turbulent boundary layer on the surface of the body. Roughness has the same effect on sphere drag, a fact that is exploited in sports by deliberate dimpling of golf balls to give them less drag at their flight $\mathrm{Re}_d \approx 10^5$. See Fig. D5.2.

Figure 5.3 is a typical experimental study of a fluid mechanics problem, aided by dimensional analysis. As time and money and demand allow, the complete three-parameter relation (5.28) could be filled out by further experiments.

### EXAMPLE 5.7

A smooth cylinder, 1 cm in diameter and 20 cm long, is tested in a wind tunnel for a crossflow of 45 m/s of air at 20°C and 1 atm. The measured drag is 2.2 ± 0.1 N. (*a*) Does this data point agree with the data in Fig. 5.3? (*b*) Can this data point be used to predict the drag of a chimney 1 m in diameter and 20 m high in winds at 20°C and 1 atm? If so, what is the recommended range of wind velocities and drag forces for this data point? (*c*) Why are the answers to part (*b*) always the same, regardless of the chimney height, as long as $L = 20d$?

### Solution

(*a*) For air at 20°C and 1 atm, take $\rho = 1.2$ kg/m$^3$ and $\mu = 1.8$ E−5 kg/(m-s). Since the test cylinder is short, $L/d = 20$, it should be compared with the tabulated value $C_D \approx 0.91$ in the table to the right of Fig. 5.3*a*. First calculate the Reynolds number of the test cylinder:

$$\mathrm{Re}_d = \frac{\rho U d}{\mu} = \frac{(1.2\ \mathrm{kg/m^3})(45\ \mathrm{m/s})(0.01\ \mathrm{m})}{1.8\mathrm{E}{-5}\ \mathrm{kg/(m - s)}} = 30{,}000$$

Yes, this is in the range $10^4 < \mathrm{Re} < 10^5$ listed in the table. Now calculate the test drag coefficient:

$$C_{D,\mathrm{test}} = \frac{F}{(1/2)\rho U^2 L d} = \frac{2.2\ \mathrm{N}}{(1/2)(1.2\ \mathrm{kg/m^3})(45\ \mathrm{m/s})^2(0.2\ \mathrm{m})(0.01\ \mathrm{m})} = 0.905$$

Yes, this is close, and certainly within the range of ±5 percent stated by the test results.

*Ans.* (*a*)

(*b*) Since the chimney has $L/d = 20$, we can use the data if the Reynolds number range is correct:

$$10^4 < \frac{(1.2\ \mathrm{kg/m^3})U_{\mathrm{chimney}}(1\ \mathrm{m})}{1.8\ \mathrm{E}{-5}\ \mathrm{kg/(m \cdot s)}} < 10^5 \quad \text{if} \quad 0.15\ \frac{\mathrm{m}}{\mathrm{s}} < U_{\mathrm{chimney}} < 1.5\ \frac{\mathrm{m}}{\mathrm{s}}$$

These are negligible winds, so the test data point is not very useful *Ans.* (*b*)

The drag forces in this range are also negligibly small:

$$F_{\min} = C_D \frac{\rho}{2} U_{\min}^2 L d = (0.91)\left(\frac{1.2\ \mathrm{kg/m^3}}{2}\right)(0.15\ \mathrm{m/s})^2(20\ \mathrm{m})(1\ \mathrm{m}) = 0.25\ \mathrm{N}$$

$$F_{\max} = C_D \frac{\rho}{2} U_{\max}^2 L d = (0.91)\left(\frac{1.2\ \mathrm{kg/m^3}}{2}\right)(1.5\ \mathrm{m/s})^2(20\ \mathrm{m})(1\ \mathrm{m}) = 25\ \mathrm{N}$$

(*c*) Try this yourself. Choose any 20:1 size for the chimney, even something silly like 20 mm:1 mm. You will get the same results for $U$ and $F$ as in part (*b*) above. This is because the product $Ud$ occurs in $\mathrm{Re}_d$ and, if $L = 20d$, the same product occurs in the drag force. For example, for $\mathrm{Re} = 10^4$,

$$Ud = 10^4 \frac{\mu}{\rho} \quad \text{then } F = C_D \frac{\rho}{2} U^2 L d = C_D \frac{\rho}{2} U^2 (20d) d = 20 C_D \frac{\rho}{2} (Ud)^2 = 20 C_D \frac{\rho}{2}\left(\frac{10^4 \mu}{\rho}\right)^2$$

The answer is always $F_{\min} = 0.25$ N. This is an algebraic quirk that seldom occurs.

### EXAMPLE 5.8

Telephone wires are said to "sing" in the wind. Consider a wire of diameter 8 mm. At what sea-level wind velocity, if any, will the wire sing a middle C note?

**Solution**

For sea-level air take $\nu \approx 1.5$ E−5 $m^2/s$. For nonmusical readers, middle C is 262 Hz. Measured shedding rates are plotted in Fig. 5.2*b*. Over a wide range, the Strouhal number is approximately 0.2, which we can take as a first guess. Note that $(\omega/2\pi) = f$, the shedding frequency. Thus

$$\text{St} = \frac{fd}{U} = \frac{(262\ \text{s}^{-1})(0.008\ \text{m})}{U} \approx 0.2$$

$$U \approx 10.5\ \frac{\text{m}}{\text{s}}$$

Now check the Reynolds number to see if we fall into the appropriate range:

$$\text{Re}_d = \frac{Ud}{\nu} = \frac{(10.5\ \text{m/s})(0.008\ \text{m})}{1.5\ \text{E}{-5}\ \text{m}^2/\text{s}} \approx 5600$$

In Fig. 5.2*b*, at Re = 5600, maybe St is a little higher, at about 0.21. Thus a slightly improved estimate is

$$U_{\text{wind}} = (262)(0.008)/(0.21) \approx 10.0\ \text{m/s} \qquad \textit{Ans.}$$

## 5.5 Modeling and Similarity

So far we have learned about dimensional homogeneity and the pi theorem method, using power products, for converting a homogeneous physical relation to dimensionless form. This is straightforward mathematically, but certain engineering difficulties need to be discussed.

First, we have more or less taken for granted that the variables that affect the process can be listed and analyzed. Actually, selection of the important variables requires considerable judgment and experience. The engineer must decide, for example, whether viscosity can be neglected. Are there significant temperature effects? Is surface tension important? What about wall roughness? Each pi group that is retained increases the expense and effort required. Judgment in selecting variables will come through practice and maturity; this book should provide some of the necessary experience.

Once the variables are selected and the dimensional analysis is performed, the experimenter seeks to achieve *similarity* between the model tested and the prototype to be designed. With sufficient testing, the model data will reveal the desired dimensionless function between variables:

$$\Pi_1 = f(\Pi_2, \Pi_3, \ldots \Pi_k) \tag{5.29}$$

With Eq. (5.29) available in chart, graphical, or analytical form, we are in a position to ensure complete similarity between model and prototype. A formal statement would be as follows:

> Flow conditions for a model test are completely similar if all relevant dimensionless parameters have the same corresponding values for the model and the prototype.

This follows mathematically from Eq. (5.29). If $\Pi_{2m} = \Pi_{2p}$, $\Pi_{3m} = \Pi_{3p}$, and so forth, Eq. (5.29) guarantees that the desired output $\Pi_{1m}$ will equal $\Pi_{1p}$. But this is easier said than done, as we now discuss. There are specialized texts on model testing [30–32].

Instead of complete similarity, the engineering literature speaks of particular types of similarity, the most common being geometric, kinematic, dynamic, and thermal. Let us consider each separately.

## Geometric Similarity

Geometric similarity concerns the length dimension $\{L\}$ and must be ensured before any sensible model testing can proceed. A formal definition is as follows:

> A model and prototype are *geometrically similar* if and only if all body dimensions in all three coordinates have the same linear scale ratio.

Note that *all* length scales must be the same. It is as if you took a photograph of the prototype and reduced it or enlarged it until it fitted the size of the model. If the model is to be made one-tenth the prototype size, its length, width, and height must each be one-tenth as large. Not only that, but also its entire shape must be one-tenth as large, and technically we speak of *homologous* points, which are points that have the same relative location. For example, the nose of the prototype is homologous to the nose of the model. The left wingtip of the prototype is homologous to the left wingtip of the model. Then geometric similarity requires that all homologous points be related by the same linear scale ratio. This applies to the fluid geometry as well as the model geometry.

> All angles are preserved in geometric similarity. All flow directions are preserved. The orientations of model and prototype with respect to the surroundings must be identical.

Figure 5.4 illustrates a prototype wing and a one-tenth-scale model. The model lengths are all one-tenth as large, but its angle of attack with respect to the free stream is the same for both model and prototype: 10° not 1°. All physical details on the model must be scaled, and some are rather subtle and sometimes overlooked:

1. The model nose radius must be one-tenth as large.
2. The model surface roughness must be one-tenth as large.

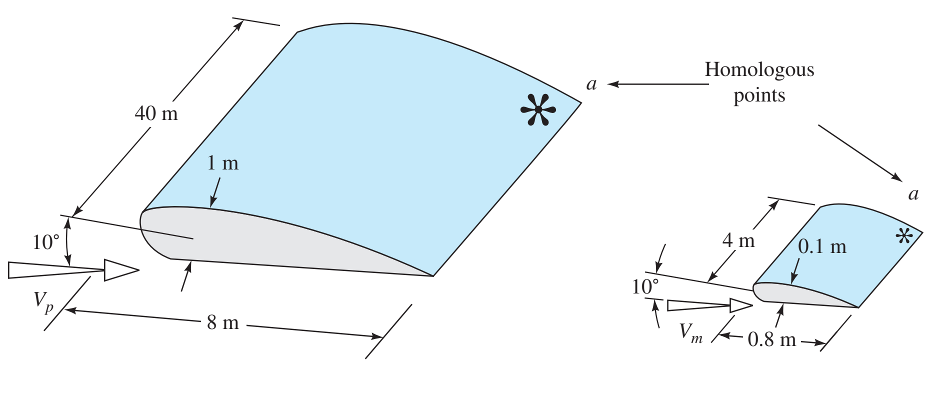



**Fig. 5.4** Geometric similarity in model testing: (*a*) prototype; (*b*) one-tenth-scale model.

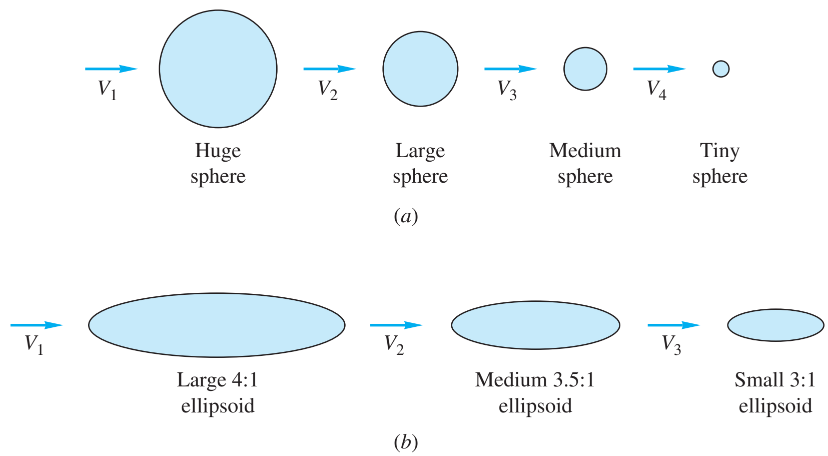



**Fig. 5.5** Geometric similarity and dissimilarity of flows: (*a*) similar; (*b*) dissimilar.

3. If the prototype has a 5-mm boundary layer trip wire 1.5 m from the leading edge, the model should have a 0.5-mm trip wire 0.15 m from its leading edge.
4. If the prototype is constructed with protruding fasteners, the model should have homologous protruding fasteners one-tenth as large.

And so on. Any departure from these details is a violation of geometric similarity and must be justified by experimental comparison to show that the prototype behavior was not significantly affected by the discrepancy.

Models that appear similar in shape but that clearly violate geometric similarity should not be compared except at your own risk. Figure 5.5 illustrates this point. The spheres in Fig. 5.5*a* are all geometrically similar and can be tested with a high expectation of success if the Reynolds number, Froude number, or the like is matched. But the ellipsoids in Fig. 5.5*b* merely *look* similar. They actually have different linear scale ratios and therefore cannot be compared in a rational manner, even though they may have identical Reynolds and Froude numbers and so on. The data will not be the same for these ellipsoids, and any attempt to “compare” them is a matter of rough engineering judgment.

## Kinematic Similarity

Kinematic similarity requires that the model and prototype have the same length scale ratio and the same time scale ratio. The result is that the velocity scale ratio will be the same for both. As Langhaar [4] states it:

> The motions of two systems are kinematically similar if homologous particles lie at homologous points at homologous times.

Length scale equivalence simply implies geometric similarity, but time scale equivalence may require additional dynamic considerations such as equivalence of the Reynolds and Mach numbers.

One special case is incompressible frictionless flow with no free surface, as sketched in Fig. 5.6*a*. These perfect-fluid flows are kinematically similar with independent length and time scales, and no additional parameters are necessary (see Chap. 8 for further details).

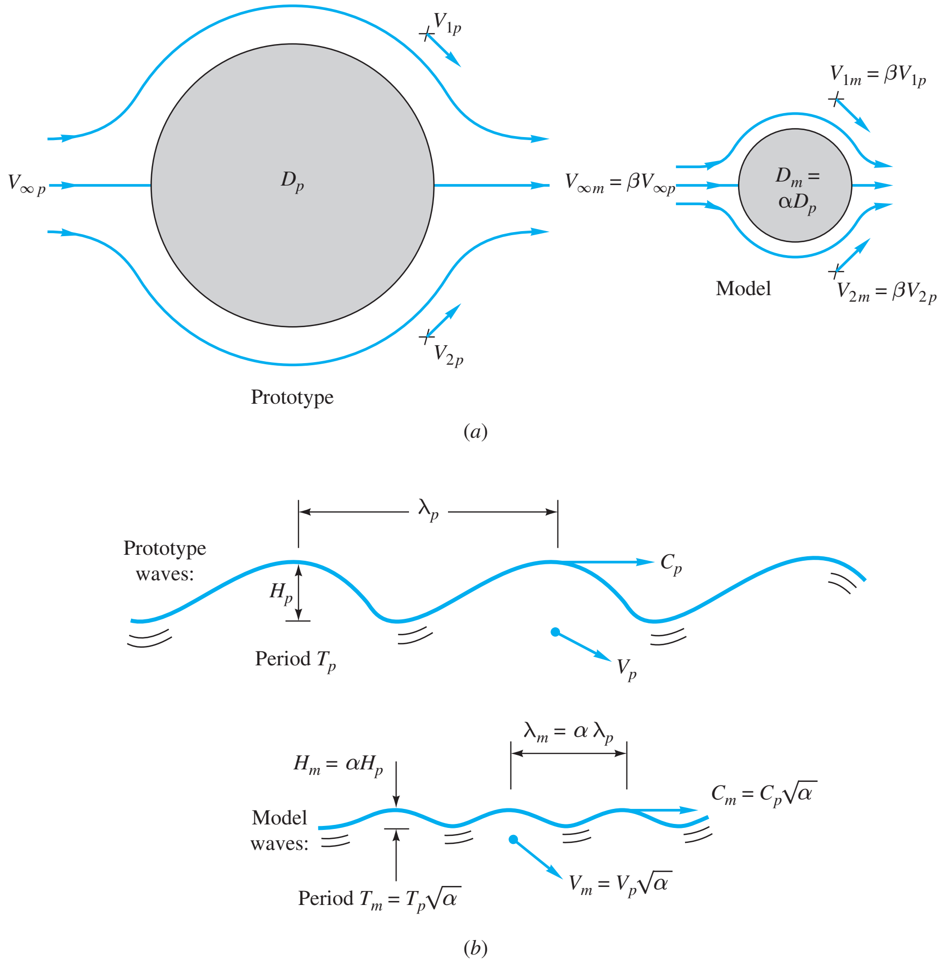



**Fig. 5.6** Frictionless low-speed flows are kinematically similar: (*a*) Flows with no free surface are kinematically similar with independent length and time scale ratios; (*b*) free-surface flows are kinematically similar with length and time scales related by the Froude number.

## Froude Scaling

Frictionless flows with a free surface, as in Fig. 5.6*b*, are kinematically similar if their Froude numbers are equal:

$$\mathrm{Fr}_m = \frac{V_m^2}{gL_m} = \frac{V_p^2}{gL_p} = Fr_p \tag{5.30}$$

Note that the Froude number contains only length and time dimensions and hence is a purely kinematic parameter that fixes the relation between length and time. From Eq. (5.30), if the length scale is

$$L_m = \alpha L_p \tag{5.31}$$

where $\alpha$ is a dimensionless ratio, the velocity scale is

$$\frac{V_m}{V_p} = \left(\frac{L_m}{L_p}\right)^{1/2} = \sqrt{\alpha} \tag{5.32}$$

and the time scale is

$$\frac{T_m}{T_p} = \frac{L_m / V_m}{L_p / V_p} = \sqrt{\alpha} \tag{5.33}$$

These Froude-scaling kinematic relations are illustrated in Fig. 5.6*b* for wave motion modeling. If the waves are related by the length scale $\alpha$, then the wave period, propagation speed, and particle velocities are related by $\sqrt{\alpha}$.

If viscosity, surface tension, or compressibility is important, kinematic similarity depends on the achievement of dynamic similarity.

## Dynamic Similarity

Dynamic similarity exists when the model and the prototype have the same length scale ratio, time scale ratio, and force scale (or mass scale) ratio. Again geometric similarity is a first requirement; without it, proceed no further. Then dynamic similarity exists, simultaneous with kinematic similarity, if the model and prototype force and pressure coefficients are identical. This is ensured if

1. For compressible flow, the model and prototype Reynolds number and Mach number and specific-heat ratio are correspondingly equal.
2. For incompressible flow
   *a.* With no free surface: model and prototype Reynolds numbers are equal.
   *b.* With a free surface: model and prototype Reynolds number, Froude number, and (if necessary) Weber number and cavitation number are correspondingly equal.

Mathematically, Newton's law for any fluid particle requires that the sum of the pressure force, gravity force, and friction force equal the acceleration term, or inertia force,

$$\mathbf{F}_p + \mathbf{F}_g + \mathbf{F}_f = \mathbf{F}_i$$

The dynamic similarity laws listed above ensure that each of these forces will be in the same ratio and have equivalent directions between model and prototype. Figure 5.7

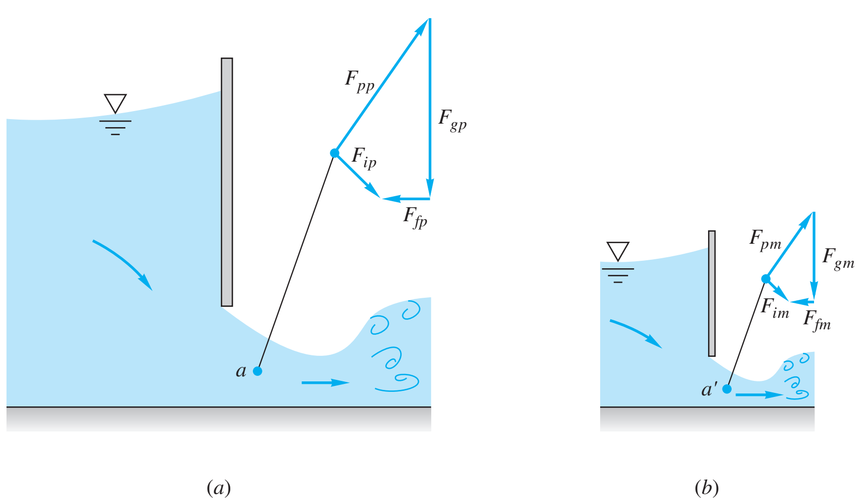



**Fig. 5.7** Dynamic similarity in sluice gate flow. Model and prototype yield identical homologous force polygons if the Reynolds and Froude numbers are the same corresponding values: (*a*) prototype; (*b*) model.

shows an example for flow through a sluice gate. The force polygons at homologous points have exactly the same shape if the Reynolds and Froude numbers are equal (neglecting surface tension and cavitation, of course). Kinematic similarity is also ensured by these model laws.

### Discrepancies in Water and Air Testing

The perfect dynamic similarity shown in Fig. 5.7 is more of a dream than a reality because true equivalence of Reynolds and Froude numbers can be achieved only by dramatic changes in fluid properties, whereas in fact most model testing is simply done with water or air, the cheapest fluids available.

First consider hydraulic model testing with a free surface. Dynamic similarity requires equivalent Froude numbers, Eq. (5.30), *and* equivalent Reynolds numbers:

$$\frac{V_m L_m}{\nu_m} = \frac{V_p L_p}{\nu_p} \tag{5.34}$$

But both velocity and length are constrained by the Froude number, Eqs. (5.31) and (5.32). Therefore, for a given length scale ratio $\alpha$, Eq. (5.34) is true only if

$$\frac{\nu_m}{\nu_p} = \frac{L_m}{L_p}\frac{V_m}{V_p} = \alpha\sqrt{\alpha} = \alpha^{3/2} \tag{5.35}$$

For example, for a one-tenth-scale model, $\alpha = 0.1$ and $\alpha^{3/2} = 0.032$. Since $\nu_p$ is undoubtedly water, we need a fluid with only 0.032 times the kinematic viscosity of water to achieve dynamic similarity. Referring to Table 1.4, we see that this is impossible: Even mercury has only one-ninth the kinematic viscosity of water, and a mercury hydraulic model would be expensive and bad for your health. In practice, water is used for both the model and the prototype, and the Reynolds number similarity (5.34) is unavoidably violated. The Froude number is held constant since it is the dominant parameter in free-surface flows. Typically the Reynolds number of the model flow is too small by a factor of 10 to 1000. As shown in Fig. 5.8, the low-Reynolds-number model data are used to estimate by extrapolation the desired high-Reynolds-number prototype data. As the figure indicates, there is obviously considerable uncertainty in using such an extrapolation, but there is no other practical alternative in hydraulic model testing.

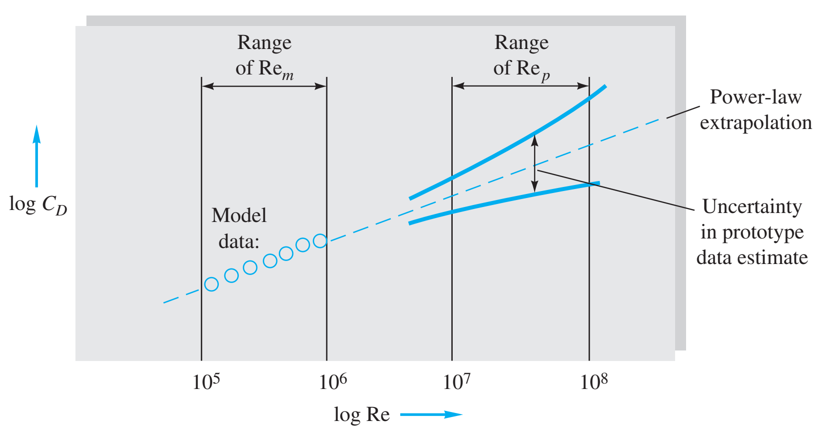



**Fig. 5.8** Reynolds-number extrapolation, or scaling, of hydraulic data with equal Froude numbers.

Second, consider aerodynamic model testing in air with no free surface. The important parameters are the Reynolds number and the Mach number. Equation (5.34) should be satisfied, plus the compressibility criterion

$$\frac{V_m}{a_m} = \frac{V_p}{a_p} \tag{5.36}$$

Elimination of $V_m/V_p$ between (5.34) and (5.36) gives

$$\frac{\nu_m}{\nu_p} = \frac{L_m}{L_p}\frac{a_m}{a_p} \tag{5.37}$$

Since the prototype is no doubt an air operation, we need a wind-tunnel fluid of low viscosity and high speed of sound. Hydrogen is the only practical example, but clearly it is too expensive and dangerous. Therefore, wind tunnels normally operate with air as the working fluid. Cooling and pressurizing the air will bring Eq. (5.37) into better agreement but not enough to satisfy a length scale reduction of, say, one-tenth. Therefore Reynolds number scaling is also commonly violated in aerodynamic testing, and an extrapolation like that in Fig. 5.8 is required here also.

There are specialized monographs devoted entirely to wind tunnel testing: low speed [38], high speed [39], and a detailed general discussion [40]. The following example illustrates modeling discrepancies in aeronautical testing.

**EXAMPLE 5.9**

A prototype airplane, with a chord length of 1.6 m, is to fly at Ma = 2 at 10 km standard altitude. A one-eighth scale model is to be tested in a helium wind tunnel at 100°C and 1 atm. Find the helium test section velocity that will match (*a*) the Mach number or (*b*) the Reynolds number of the prototype. In each case criticize the lack of dynamic similarity. (*c*) What high pressure in the helium tunnel will match *both* the Mach and Reynolds numbers? (*d*) Why does part (*c*) *still* not achieve dynamic similarity?

**Solution**

For helium, from Table A.4, $R = 2077\ \text{m}^2/(\text{s}^2\text{-K})$, $k = 1.66$, and estimate $\mu_{He} \approx 2.32$ E−5 kg/(m · s) from the power-law, $n = 0.67$, in the table. (*a*) Calculate the helium speed of sound and velocity:

$$a_{He} = \sqrt{(kRT)_{He}} = \sqrt{(1.66)(2077\ \text{m}^2/\text{s}^2\text{K}) \times (373\ \text{K})} = 1134\ \text{m/s}$$

$$\text{Ma}_{air} = \text{Ma}_{He} = 2.0 = \frac{V_{He}}{a_{He}} = \frac{V_{He}}{1134\ \text{m/s}}$$

$$V_{He} = 2268\ \frac{\text{m}}{\text{s}} \qquad \textit{Ans. (a)}$$

For dynamic similarity, the Reynolds numbers should also be equal. From Table A.6 at an altitude of 10,000 m, read $\rho_{air} = 0.4125\ \text{kg/m}^3$, $a_{air} = 299.5$ m/s, and estimate $\mu_{air} \approx 1.48$ E−5 kg/m · s from the power-law, $n = 0.7$, in Table A.4. The air velocity is $V_{air} = (\text{Ma})(a_{air}) = 2(299.5) = 599$ m/s. The model chord length is (1.6 m)/8 = 0.2 m. The helium density is $\rho_{He} = (p/RT)_{He} = (101{,}350 \text{ Pa})/[(2077 \text{ m}^2/\text{s}^2 \text{ K})(373 \text{ K})] = 0.131 \text{ kg/m}^3$. Now calculate the two Reynolds numbers:

$$\mathrm{Re}_{C,\text{air}} = \frac{\rho VC}{\mu}\bigg|_{\text{air}} = \frac{(0.4125 \text{ kg/m}^3)(599 \text{ m/s})(1.6 \text{ m})}{1.48 \text{ E}{-5} \text{ kg/(m}\cdot\text{s)}} = 26.6 \text{ E6}$$

$$\mathrm{Re}_{C,\text{He}} = \frac{\rho VC}{\mu}\bigg|_{\text{He}} = \frac{(0.131 \text{ kg/m}^3)(2268 \text{ m/s})(0.2 \text{ m})}{2.32 \text{ E}{-5} \text{ kg/(m}\cdot\text{s)}} = 2.56 \text{ E6}$$

The model Reynolds number is 10 times less than the prototype. This is typical when using small-scale models. The test results must be extrapolated for Reynolds number effects.
(*b*) Now ignore Mach number and let the model Reynolds number match the prototype:

$$\mathrm{Re}_{He} = \mathrm{Re}_{air} = 26.6 \text{ E6} = \frac{(0.131 \text{ kg/m}^3)V_{He}(0.2 \text{ m})}{2.32 \text{ E}{-5} \text{ kg/(m}\cdot\text{s)}}$$

$$V_{He} = 23{,}600 \frac{\text{m}}{\text{s}} \qquad \textit{Ans. (b)}$$

This is ridiculous: a hypersonic Mach number of 21, suitable for escaping from the earth's gravity. One should match the Mach numbers and correct for a lower Reynolds number.
(*c*) Match both Reynolds and Mach numbers by increasing the helium density:
Ma matches if

$$V_{He} = 2268 \frac{\text{m}}{\text{s}}$$

Then

$$\mathrm{Re}_{He} = 26.6 \text{ E6} = \frac{\rho_{He}(2268 \text{ m/s})(0.2 \text{ m})}{2.32 \text{ E}{-5} \text{ kg/(m}\cdot\text{s)}}$$

Solve for

$$\rho_{He} = 1.36 \frac{\text{kg}}{\text{m}^3} \quad p_{He} = \rho RT|_{He} = (1.36)(2077)(373) = 1.05 \text{ E6 Pa} \qquad \textit{Ans. (c)}$$

A match is possible if we increase the tunnel pressure by a factor of ten, a daunting task.
(*d*) Even with Ma and Re matched, we are *still* not dynamically similar because the two gases have different specific heat ratios: $k_{He} = 1.66$ and $k_{air} = 1.40$. This discrepancy will cause substantial differences in pressure, density, and temperature throughout supersonic flow.

Figure 5.9 shows a hydraulic model of the Bluestone Lake Dam in West Virginia. The model itself is located at the U.S. Army Waterways Experiment Station in Vicksburg, MS. The horizontal scale is 1:65, which is sufficient that the vertical scale can also be 1:65 without incurring significant surface tension (Weber number) effects. Velocities are scaled by the Froude number. However, the prototype Reynolds number, which is of order 1 E7, cannot be matched here. The engineers set the Reynolds number at about 2 E4, high enough for a reasonable approximation of prototype turbulent flow viscous effects. Note the intense turbulence below the dam. The downstream bed, or *apron,* of a dam must be strengthened structurally to avoid bed erosion.

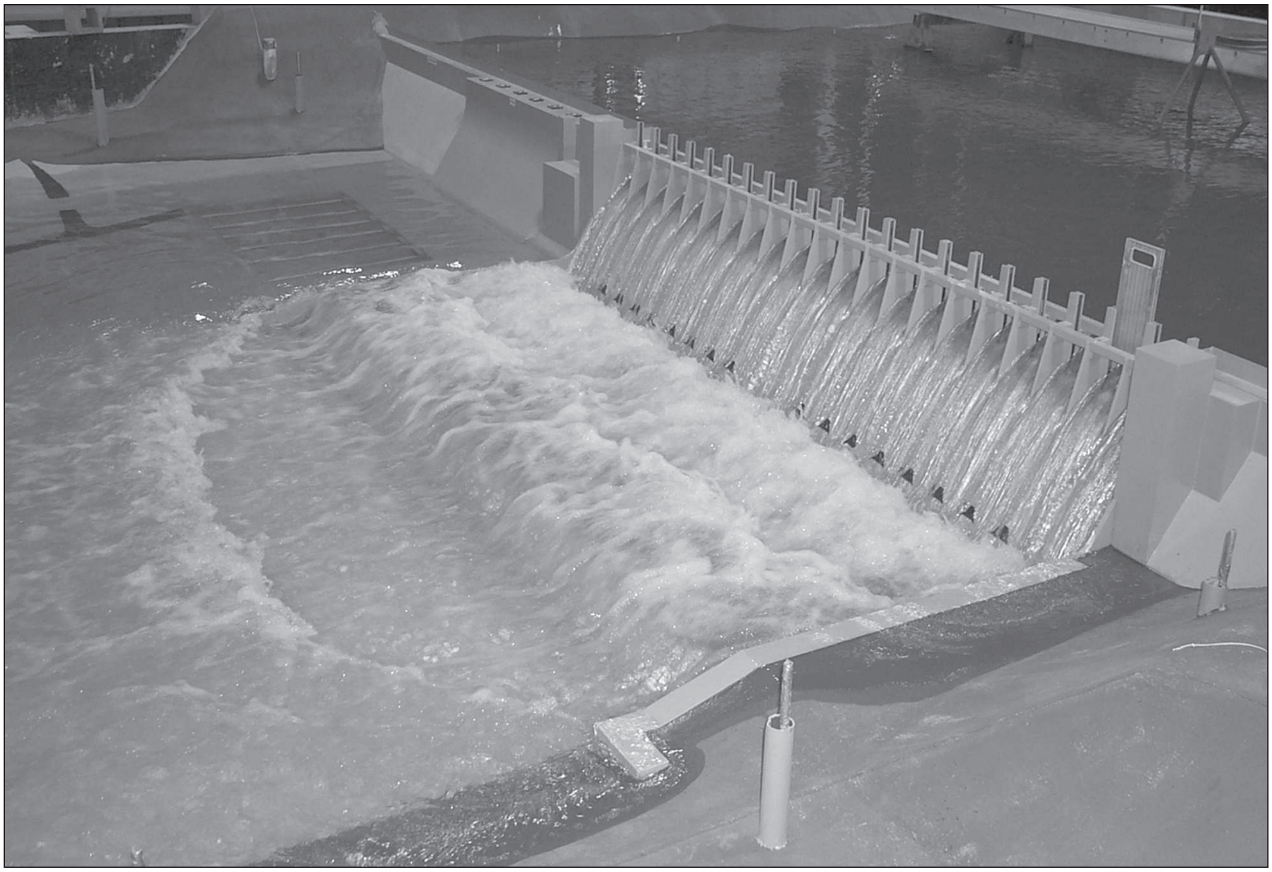


**Fig. 5.9** Hydraulic model of the Bluestone Lake Dam on the New River near Hinton, West Virginia. The model scale is 1:65 both vertically and horizontally, and the Reynolds number, though far below the prototype value, is set high enough for the flow to be turbulent. *(Courtesy of the U.S. Army Corps of Engineers Waterways Experiment Station.)*

For hydraulic models of larger scale, such as harbors, estuaries, and embayments, geometric similarity may be violated of necessity. The vertical scale will be distorted to avoid Weber number effects. For example, the horizontal scale may be 1:1000, while the vertical scale is only 1:100. Thus the model channel may be *deeper* relative to its horizontal dimensions. Since deeper passages flow more efficiently, the model channel bottom may be deliberately roughened to create the friction level expected in the prototype.

### EXAMPLE 5.10

The pressure drop due to friction for flow in a long, smooth pipe is a function of average flow velocity, density, viscosity, and pipe length and diameter: $\Delta p = \text{fcn}(V, \rho, \mu, L, D)$. We wish to know how $\Delta p$ varies with $V$. ($a$) Use the pi theorem to rewrite this function in dimensionless form. (*b*) Then plot this function, using the following data for three pipes and three fluids:

| $D$, cm | $L$, m | $Q$, $m^3/h$ | $\Delta p$, Pa | $\rho$, $kg/m^3$ | $\mu$, kg/(m · s) | $V$, m/s* |
|---|---|---|---|---|---|---|
| 1.0 | 5.0 | 0.3 | 4,680 | 680† | 2.92 E-4† | 1.06 |
| 1.0 | 7.0 | 0.6 | 22,300 | 680† | 2.92 E-4† | 2.12 |
| 1.0 | 9.0 | 1.0 | 70,800 | 680† | 2.92 E-4† | 3.54 |
| 2.0 | 4.0 | 1.0 | 2,080 | 998‡ | 0.0010‡ | 0.88 |
| 2.0 | 6.0 | 2.0 | 10,500 | 998‡ | 0.0010‡ | 1.77 |
| 2.0 | 8.0 | 3.1 | 30,400 | 998‡ | 0.0010‡ | 2.74 |
| 3.0 | 3.0 | 0.5 | 540 | 13,550§ | 1.56 E-3§ | 0.20 |
| 3.0 | 4.0 | 1.0 | 2,480 | 13,550§ | 1.56 E-3§ | 0.39 |
| 3.0 | 5.0 | 1.7 | 9,600 | 13,550§ | 1.56 E-3§ | 0.67 |

*$V = Q/A$, $A = \pi D^2/4$.
†Gasoline.
‡Water.
§Mercury.

(*c*) Suppose it is further known that $\Delta p$ is proportional to $L$ (which is quite true for long pipes with well-rounded entrances). Use this information to simplify and improve the pi theorem formulation. Plot the dimensionless data in this improved manner and comment on the results.

## Solution

There are six variables with three primary dimensions involved $\{MLT\}$. Therefore, we expect that $j = 6 - 3 = 3$ pi groups. We are correct, for we can find three variables that do not form a pi product (e.g., $\rho$, $V$, $L$). Carefully select three ($j$) repeating variables, but not including $\Delta p$ or $V$, which we plan to plot versus each other. We select ($\rho$, $\mu$, $D$), and the pi theorem guarantees that three independent power-product groups will occur:

$$\Pi_1 = \rho^a \mu^b D^c\, \Delta p \qquad \Pi_2 = \rho^d \mu^e D^f V \qquad \Pi_3 = \rho^g \mu^h D^i L$$

or

$$\Pi_1 = \frac{\rho D^2 \Delta p}{\mu^2} \qquad \Pi_2 = \frac{\rho V D}{\mu} \qquad \Pi_3 = \frac{L}{D}$$

We have omitted the algebra of finding ($a$, $b$, $c$, $d$, $e$, $f$, $g$, $h$, $i$) by setting all exponents to zero $M^0$, $L^0$, $T^0$. Therefore, we wish to plot the dimensionless relation

$$\frac{\rho D^2\, \Delta p}{\mu^2} = \text{fcn}\left(\frac{\rho V D}{\mu}, \frac{L}{D}\right) \qquad \textit{Ans. (a)}$$

We plot $\Pi_1$ versus $\Pi_2$ with $\Pi_3$ as a parameter. There will be nine data points. For example, the first row in the data here yields

$$\frac{\rho D^2\, \Delta p}{\mu^2} = \frac{(680)(0.01)^2(4680)}{(2.92\text{ E-4})^2} = 3.73\text{ E9}$$

$$\frac{\rho V D}{\mu} = \frac{(680)(1.06)(0.01)}{2.92\text{ E-4}} = 24{,}700 \qquad \frac{L}{D} = 500$$

The nine data points are plotted as the open circles in Fig. 5.10. The values of $L/D$ are listed for each point, and we see a significant length effect. In fact, if we connect the only two points that have the same $L/D$ (= 200), we could see (and cross-plot to verify) that $\Delta p$ increases linearly with $L$, as stated in the last part of the problem. Since $L$ occurs only in

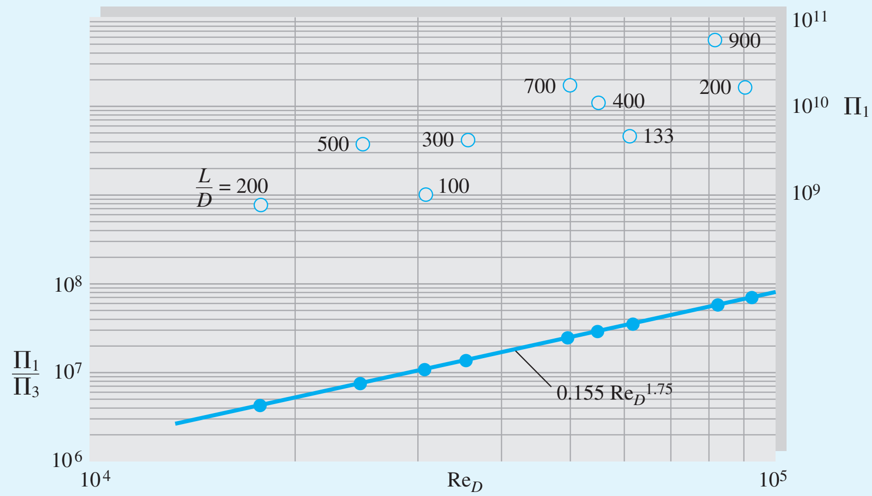



**Fig. 5.10** Two different correlations of the data in Example 5.10: Open circles when plotting $\rho D^2\,\Delta p/\mu^2$ versus $\mathrm{Re}_D$, $L/D$ is a parameter; once it is known that $\Delta p$ is proportional to $L$, a replot (solid circles) of $\rho D^3\,\Delta p/(L\mu^2)$ versus $\mathrm{Re}_D$ collapses into a single power-law curve.

$\Pi_3 = L/D$, the function $\Pi_1 = \mathrm{fcn}(\Pi_2, \Pi_3)$ must reduce to $\Pi_1 = (L/D)\,\mathrm{fcn}(\Pi_2)$, or simply a function involving only *two* parameters:

$$\frac{\rho D^3\,\Delta p}{L\mu^2} = \mathrm{fcn}\left(\frac{\rho V D}{\mu}\right) \qquad \text{flow in a long pipe} \qquad \textit{Ans. (c)}$$

We now modify each data point in Fig. 5.10 by dividing it by its $L/D$ value. For example, for the first row of data, $\rho D^3\,\Delta p/(L\mu^2) = (3.73\text{ E9})/500 = 7.46\text{ E6}$. We replot these new data points as solid circles in Fig. 5.10. They correlate almost perfectly into a straight-line power-law function:

$$\frac{\rho D^3\,\Delta p}{L\mu^2} \approx 0.155\left(\frac{\rho V D}{\mu}\right)^{1.75} \qquad \textit{Ans. (c)}$$

All newtonian smooth pipe flows should correlate in this manner. This example is a variation of the first completely successful dimensional analysis, pipe-flow friction, performed by Prandtl's student Paul Blasius, who published a related plot in 1911. For this range of (turbulent flow) Reynolds numbers, the pressure drop increases approximately as $V^{1.75}$.

## EXAMPLE 5.11

The smooth sphere data plotted in Fig. 5.3*a* represent dimensionless drag versus dimensionless *viscosity*, since ($\rho$, $V$, $d$) were selected as scaling or repeating variables. (*a*) Replot these data to display the effect of dimensionless *velocity* on the drag. (*b*) Use your new figure to predict the terminal (zero-acceleration) velocity of a 1-cm-diameter steel ball (SG = 7.86) falling through water at 20°C.

### Solution

- *Assumptions:* Fig 5.3*a* is valid for any smooth sphere in that Reynolds number range.
- *Approach (a):* Form pi groups from the function $F = \mathrm{fcn}(d, V, \rho, \mu)$ in such a way that $F$ is plotted versus $V$. The answer was already given as Eq. (5.16), but let us review the

steps. The proper scaling variables are ($\rho$, $\mu$, $d$), which do *not* form a pi. Therefore $j = 3$, and we expect $n - j = 5 - 3 = 2$ pi groups. Skipping the algebra, they arise as follows:

$$\Pi_1 = \rho^a \mu^b d^c F = \frac{\rho F}{\mu^2} \qquad \Pi_2 = \rho^a \mu^b d^c V = \frac{\rho V d}{\mu} \qquad \textit{Ans. (a)}$$

We may replot the data of Fig. 5.3*a* in this new form, noting that $\Pi_1 \equiv (\pi/8)(C_D)(\mathrm{Re})^2$. This replot is shown as Fig. 5.11. The drag increases rapidly with velocity up to transition, where there is a slight drop, after which it increases more than ever. If force is known, we may predict velocity from the figure, and vice versa.

- *Property values for part (b):* $\rho_{water} = 998\ \mathrm{kg/m^3}$ $\quad \mu_{water} = 0.001\ \mathrm{kg/(m\text{-}s)}$

$$\rho_{steel} = 7.86\rho_{water} = 7844\ \mathrm{kg/m^3}.$$

- *Solution to part (b):* For terminal velocity, the drag force equals the net weight of the sphere in water:

$$F = W_{net} = (\rho_s - \rho_w) g \frac{\pi}{6} d^3 = (7840 - 998)(9.81)\left(\frac{\pi}{6}\right)(0.01)^3 = 0.0351\ \mathrm{N}$$

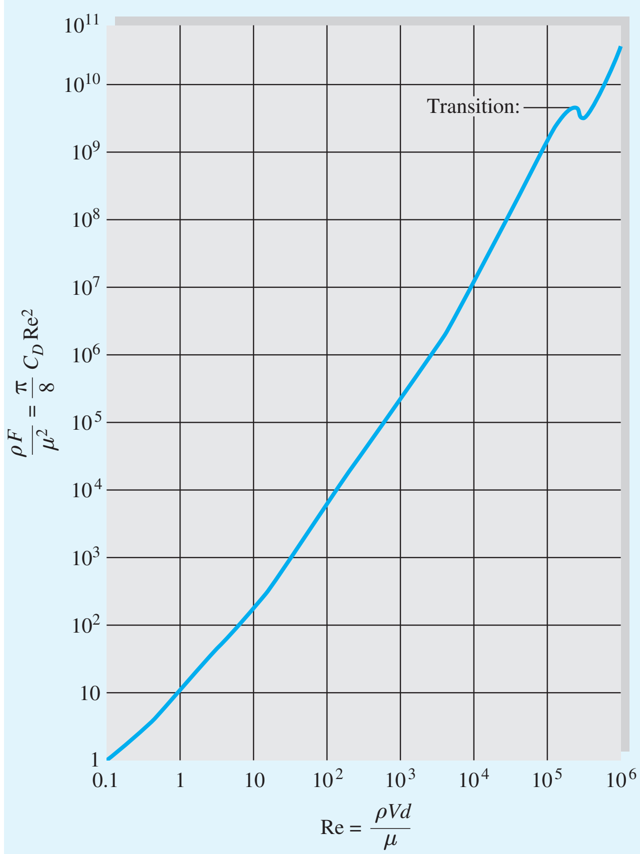



**Fig. 5.11** Cross-plot of sphere drag data from Fig. 5.3*a* to show dimensionless force versus dimensionless velocity.

Therefore, the ordinate of Fig. 5.11 is known:

$$\text{Falling steel sphere:} \qquad \frac{\rho F}{\mu^2} = \frac{(998 \text{ kg/m}^3)(0.0351 \text{ N})}{[0.001 \text{ kg/(m} \cdot \text{s)}]^2} \approx 3.5 \text{ E7}$$

From Fig. 5.11, at $\rho F/\mu^2 \approx 3.5$ E7, a magnifying glass reveals that $\text{Re}_d \approx 2$ E4. Then a crude estimate of the terminal fall velocity is

$$\frac{\rho V d}{\mu} \approx 20{,}000 \qquad \text{or} \qquad V \approx \frac{20{,}000[0.001 \text{ kg/(m} \cdot \text{s)}]}{(998 \text{ kg/m}^3)(0.01 \text{ m})} \approx 2.0 \frac{\text{m}}{\text{s}} \qquad \textit{Ans. (b)}$$

- *Comments:* Better accuracy could be obtained by expanding the scale of Fig. 5.11 in the region of the given force coefficient. However, there is considerable uncertainty in published drag data for spheres, so the predicted fall velocity is probably uncertain by at least ±10 percent.

  Note that we found the answer directly from Fig. 5.11. We could use Fig. 5.3*a* also but would have to iterate between the ordinate and abscissa to obtain the final result, since *V* is contained in both plotted variables.

## Summary

Chapters 3 and 4 presented integral and differential methods of mathematical analysis of fluid flow. This chapter introduces the third and final method: experimentation, as supplemented by the technique of dimensional analysis. Tests and experiments are used both to strengthen existing theories and to provide useful engineering results when theory is inadequate.

The chapter begins with a discussion of some familiar physical relations and how they can be recast in dimensionless form because they satisfy the principle of dimensional homogeneity. A general technique, the pi theorem, is then presented for systematically finding a set of dimensionless parameters by grouping a list of variables that govern any particular physical process. A second technique, Ipsen's method, is also described. Alternately, direct application of dimensional analysis to the basic equations of fluid mechanics yields the fundamental parameters governing flow patterns: Reynolds number, Froude number, Prandtl number, Mach number, and others.

It is shown that model testing in air and water often leads to scaling difficulties for which compromises must be made. Many model tests do not achieve true dynamic similarity. The chapter ends by pointing out that classic dimensionless charts and data can be manipulated and recast to provide direct solutions to problems that would otherwise be quite cumbersome and laboriously iterative.

## Problems

Most of the problems herein are fairly straightforward. More difficult or open-ended assignments are labeled with an asterisk. Problems labeled with a computer icon may require the use of a computer. The standard end-of-chapter problems P5.1 to P5.91 (categorized in the problem list here) are followed by word problems W5.1 to W5.10, fundamentals of engineering exam problems FE5.1 to FE5.12, comprehensive applied problems C5.1 to C5.5, and design projects D5.1 and D5.2.

**Problem Distribution**

| Section | Topic | Problems |
|---|---|---|
| 5.1 | Introduction | P5.1–P5.9 |
| 5.2 | The principle of dimensional homogeneity | P5.10–P5.13 |
| 5.3 | The pi theorem; Ipsen's method | P5.14–P5.42 |
| 5.4 | Nondimensionalizing the basic equations | P5.43–P5.47 |
| 5.4 | Data for spheres, cylinders, other bodies | P5.48–P5.59 |
| 5.5 | Scaling of model data | P5.60–P5.74 |
| 5.5 | Froude and Mach number scaling | P5.75–P5.84 |
| 5.5 | Inventive rescaling of the data | P5.85–P5.91 |

## Introduction; dynamic similarity

**P5.1** For axial flow through a circular tube, the Reynolds number for transition to turbulence is approximately 2300 [see Eq. (6.2)], based on the diameter and average velocity. If $d = 5$ cm and the fluid is kerosene at 20°C, find the volume flow rate in $m^3$/h that causes transition.

**P5.2** A prototype automobile is designed for cold weather in Denver, CO (−10°C, 83 kPa). Its drag force is to be tested on a one-seventh-scale model in a wind tunnel at 150 mi/h, 20°C, and 1 atm. If the model and prototype are to satisfy dynamic similarity, what prototype velocity, in mi/h, needs to be matched? Comment on your result.

**P5.3** The transfer of energy by viscous dissipation is dependent upon viscosity $\mu$, thermal conductivity $k$, stream velocity $U$, and stream temperature $T_0$. Group these quantities, if possible, into the dimensionless *Brinkman number,* which is proportional to $\mu$.

**P5.4** When tested in water at 20°C flowing at 2 m/s, an 8-cm-diameter sphere has a measured drag of 5 N. What will be the velocity and drag force on a 1.5-m-diameter weather balloon moored in sea-level standard air under dynamically similar conditions?

**P5.5** An automobile has a characteristic length and area of 8 ft and 60 $ft^2$, respectively. When tested in sea-level standard air, it has the following measured drag force versus speed:

| V, mi/h | 20 | 40 | 60 |
|---|---|---|---|
| Drag, lbf | 31 | 115 | 249 |

The same car travels in Colorado at 65 mi/h at an altitude of 3500 m. Using dimensional analysis, estimate (*a*) its drag force and (*b*) the horsepower required to overcome air drag.

**P5.6** The disk-gap-band parachute in the chapter-opener photo had a drag of 1600 lbf when tested at 15 mi/h in air at 20°C and 1 atm. (*a*) What was its drag coefficient? (*b*) If, as stated, the drag on Mars is 65,000 lbf and the velocity is 375 mi/h in the thin Mars atmosphere, $\rho \approx 0.020$ kg/$m^3$, what is the drag coefficient on Mars? (*c*) Can you explain the difference between (*a*) and (*b*)?

**P5.7** A body is dropped on the moon ($g = 1.62$ m/$s^2$) with an initial velocity of 12 m/s. By using option 2 variables, Eq. (5.11), the ground impact occurs at $t^{**} = 0.34$ and $S^{**} = 0.84$. Estimate (*a*) the initial displacement, (*b*) the final displacement, and (*c*) the time of impact.

**P5.8** The Archimedes number, Ar, used in the flow of stratified fluids, is a dimensionless combination of gravity $g$, density difference $\Delta\rho$, fluid width $L$, and viscosity $\mu$. Find the form of this number if it is proportional to $g$.

**P5.9** The *Richardson number,* Ri, which correlates the production of turbulence by buoyancy, is a dimensionless combination of the acceleration of gravity $g$, the fluid temperature $T_0$, the local temperature gradient $\partial T/\partial z$, and the local velocity gradient $\partial u/\partial z$. Determine the form of the Richardson number if it is proportional to $g$.

## The principle of dimensional homogeneity

**P5.10** Determine the dimension $\{MLT\Theta\}$ of the following quantities:

$$(a)\ \rho u \frac{\partial u}{\partial x} \quad (b)\ \int_1^2 (p - p_0)\, dA \quad (c)\ \rho c_p \frac{\partial^2 T}{\partial x\, \partial y}$$

$$(d)\ \iiint \rho \frac{\partial u}{\partial t}\, dx\, dy\, dz$$

All quantities have their standard meanings; for example, $\rho$ is density.

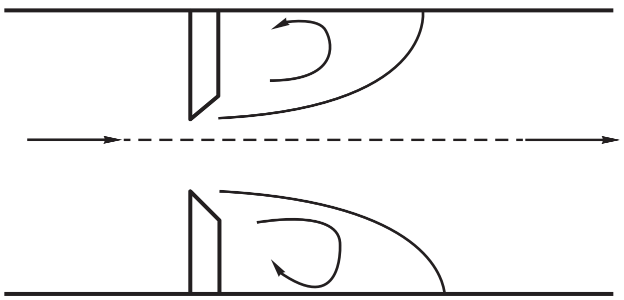
**P5.11** During World War II, Sir Geoffrey Taylor, a British fluid dynamicist, used dimensional analysis to estimate the wave speed of an atomic bomb explosion. He assumed that the blast wave radius $R$ was a function of energy released $E$, air density $\rho$, and time $t$. Use dimensional reasoning to show how wave radius must vary with time.

**P5.12** The *Stokes number,* St, used in particle dynamics studies, is a dimensionless combination of *five* variables: acceleration of gravity $g$, viscosity $\mu$, density $\rho$, particle velocity $U$, and particle diameter $D$. (*a*) If St is proportional to $\mu$ and inversely proportional to $g$, find its form. (*b*) Show that St is actually the quotient of two more traditional dimensionless groups.

**P5.13** The speed of propagation $C$ of a capillary wave in deep water is known to be a function only of density $\rho$, wavelength $\lambda$, and surface tension $\Upsilon$. Find the proper functional relationship, completing it with a dimensionless constant. For a given density and wavelength, how does the propagation speed change if the surface tension is doubled?

## The pi theorem or Ipsen's method

**P5.14** Flow in a pipe is often measured with an orifice plate, as in Fig. P5.14. The volume flow $Q$ is a function of the pressure drop $\Delta p$ across the plate, the fluid density $\rho$, the pipe diameter $D$, and the orifice diameter $d$. Rewrite this functional relationship in dimensionless form.



**P5.14**

**P5.15** The wall shear stress $\tau_w$ in a boundary layer is assumed to be a function of stream velocity $U$, boundary layer thickness $\delta$, local turbulence velocity $u'$, density $\rho$, and local pressure gradient $dp/dx$. Using ($\rho$, $U$, $\delta$) as repeating variables, rewrite this relationship as a dimensionless function.

**P5.16** Convection heat transfer data are often reported as a *heat transfer coefficient h*, defined by

$$\dot{Q} = hA\,\Delta T$$

where $\dot{Q}$ = heat flow, J/s
$A$ = surface area, $m^2$
$\Delta T$ = temperature difference, K

The dimensionless form of $h$, called the *Stanton number,* is a combination of $h$, fluid density $\rho$, specific heat $c_p$, and flow velocity $V$. Derive the Stanton number if it is proportional to $h$. What are the units of $h$?

**P5.17** If you disturb a tank of length $L$ and water depth $h$, the surface will oscillate back and forth at frequency $\Omega$, assumed here to depend also upon water density $\rho$ and the acceleration of gravity $g$. (*a*) Rewrite this as a dimensionless function. (*b*) If a tank of water sloshes at 2.0 Hz on earth, how fast would it oscillate on Mars ($g \approx 3.7\ m/s^2$)?

**P5.18** Under laminar conditions, the volume flow $Q$ through a small triangular-section pore of side length $b$ and length $L$ is a function of viscosity $\mu$, pressure drop per unit length $\Delta p/L$, and $b$. Using the pi theorem, rewrite this relation in dimensionless form. How does the volume flow change if the pore size $b$ is doubled?

**P5.19** The period of oscillation $T$ of a water surface wave is assumed to be a function of density $\rho$, wavelength $l$, depth $h$, gravity $g$, and surface tension $\Upsilon$. Rewrite this relationship in dimensionless form. What results if $\Upsilon$ is negligible? *Hint:* Take $l$, $\rho$, and $g$ as repeating variables.

**P5.20** A fixed cylinder of diameter $D$ and length $L$, immersed in a stream flowing normal to its axis at velocity $U$, will experience zero average lift. However, if the cylinder is rotating at angular velocity $\Omega$, a lift force $F$ will arise. The fluid density $\rho$ is important, but viscosity is secondary and can be neglected. Formulate this lift behavior as a dimensionless function.

**P5.21** In Example 5.1 we used the pi theorem to develop Eq. (5.2) from Eq. (5.1). Instead of merely listing the primary dimensions of each variable, some workers list the *powers* of each primary dimension for each variable in an array:

$$\begin{array}{c} \\ M \\ L \\ T \end{array}\begin{array}{c} \begin{array}{ccccc} F & L & U & \rho & \mu \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & -3 & -1 \\ -2 & 0 & -1 & 0 & -1 \end{bmatrix} \end{array}$$

This array of exponents is called the *dimensional matrix* for the given function. Show that the *rank* of this matrix (the size of the largest nonzero determinant) is equal to $j = n - k$, the desired reduction between original variables and the pi groups. This is a general property of dimensional matrices, as noted by Buckingham [1].

**P5.22** As will be discussed in Chap. 11, the power $P$ developed by a wind turbine is a function of diameter $D$, air density $\rho$, wind speed $V$, and rotation rate $\omega$. Viscosity effects are negligible. Rewrite this relationship in dimensionless form.

**P5.23** The period $T$ of vibration of a beam is a function of its length $L$, area moment of inertia $I$, modulus of elasticity $E$, density $\rho$, and Poisson's ratio $\sigma$. Rewrite this relation in dimensionless form. What further reduction can we make if $E$ and $I$ can occur only in the product form $EI$? *Hint:* Take $L$, $\rho$, and $E$ as repeating variables.

**P5.24** The lift force $F$ on a missile is a function of its length $L$, velocity $V$, diameter $D$, angle of attack $\alpha$, density $\rho$, viscosity $\mu$, and speed of sound $a$ of the air. Write out the dimensional matrix of this function and determine its rank. (See Prob. P5.21 for an explanation of this concept.) Rewrite the function in terms of pi groups.

**P5.25** The thrust $F$ of a propeller is generally thought to be a function of its diameter $D$ and angular velocity $\Omega$, the forward speed $V$, and the density $\rho$ and viscosity $\mu$ of the fluid. Rewrite this relationship as a dimensionless function.

**P5.26** A pendulum has an oscillation period $T$ which is assumed to depend on its length $L$, bob mass $m$, angle of swing $\theta$, and the acceleration of gravity. A pendulum 1 m long, with a bob mass of 200 g, is tested on earth and found to have a period of 2.04 s when swinging at 20°. (*a*) What is its period when it swings at 45°? A similarly constructed pendulum, with $L = 30$ cm and $m = 100$ g, is to swing on the moon ($g = 1.62$ m/s$^2$) at $\theta = 20°$. (*b*) What will be its period?

**P5.27** In studying sand transport by ocean waves, A. Shields in 1936 postulated that the threshold wave-induced bottom shear stress $\tau$ required to move particles depends on gravity $g$, particle size $d$ and density $\rho_p$, and water density $\rho$ and viscosity $\mu$. Find suitable dimensionless groups of this problem, which resulted in 1936 in the celebrated Shields sand transport diagram.

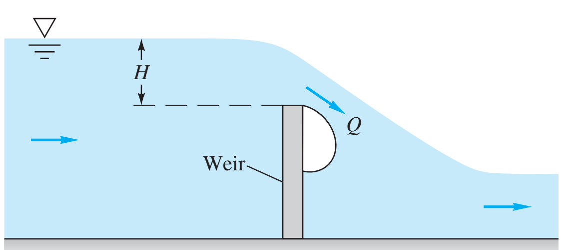
**P5.28** A simply supported beam of diameter $D$, length $L$, and modulus of elasticity $E$ is subjected to a fluid crossflow of velocity $V$, density $\rho$, and viscosity $\mu$. Its center deflection $\delta$ is assumed to be a function of all these variables. (*a*) Rewrite this proposed function in dimensionless form. (*b*) Suppose it is known that $\delta$ is independent of $\mu$, inversely proportional to $E$, and dependent only on $\rho V^2$, not $\rho$ and $V$ separately. Simplify the dimensionless function accordingly. *Hint:* Take $L$, $\rho$, and $V$ as repeating variables.

**P5.29** When fluid in a pipe is accelerated linearly from rest, it begins as laminar flow and then undergoes transition to turbulence at a time $t_{tr}$ that depends on the pipe diameter $D$, fluid acceleration $a$, density $\rho$, and viscosity $\mu$. Arrange this into a dimensionless relation between $t_{tr}$ and $D$.

**P5.30** When a large tank of high-pressure gas discharges through a nozzle, the exit mass flow $\dot{m}$ is a function of tank pressure $p_0$ and temperature $T_0$, gas constant $R$, specific heat $c_p$, and nozzle diameter $D$. Rewrite this as a dimensionless function. Check to see if you can use ($p_0$, $T_0$, $R$, $D$) as repeating variables.

**P5.31** The pressure drop per unit length in horizontal pipe flow, $\Delta p/L$, depends on the fluid density $\rho$, viscosity $\mu$, diameter $D$, and volume flow rate $Q$. Rewrite this function in terms of pi groups.

**P5.32** A *weir* is an obstruction in a channel flow that can be calibrated to measure the flow rate, as in Fig. P5.32. The volume flow $Q$ varies with gravity $g$, weir width $b$ into the paper, and upstream water height $H$ above the weir crest. If it is known that $Q$ is proportional to $b$, use the pi theorem to find a unique functional relationship $Q(g, b, H)$.

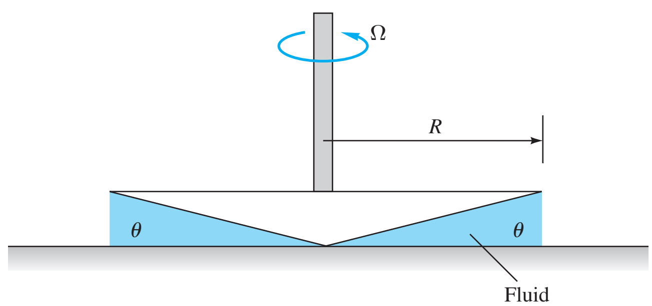



**P5.32**

**P5.33** A spar buoy (see Prob. P2.113) has a period $T$ of vertical (heave) oscillation that depends on the waterline cross-sectional area $A$, buoy mass $m$, and fluid specific weight $\gamma$. How does the period change due to doubling of (*a*) the mass and (*b*) the area? Instrument buoys should have long periods to avoid wave resonance. Sketch a possible long-period buoy design.

**P5.34** To good approximation, the thermal conductivity $k$ of a gas (see Ref. 21 of Chap. 1) depends only on the density $\rho$, mean free path $l$, gas constant $R$, and absolute temperature $T$. For air at 20°C and 1 atm, $k \approx 0.026$ W/(m · K) and $l \approx 6.5$ E-8 m. Use this information to determine $k$ for hydrogen at 20°C and 1 atm if $l \approx 1.2$ E-7 m.

**P5.35** The torque $M$ required to turn the cone-plate viscometer in Fig. P5.35 depends on the radius $R$, rotation rate $\Omega$, fluid viscosity $\mu$, and cone angle $\theta$. Rewrite this relation in dimensionless form. How does the relation simplify it if it is known that $M$ is proportional to $\theta$?




**P5.35**

**P5.36** The rate of heat loss $\dot{Q}_{loss}$ through a window or wall is a function of the temperature difference between inside and outside $\Delta T$, the window surface area $A$, and the $R$ value of the window, which has units of (ft$^2$ · h · °F)/ Btu. (*a*) Using the Buckingham Pi Theorem, find an expression for rate of heat loss as a function of the other three parameters in the problem. (*b*) If the temperature difference $\Delta T$ doubles, by what factor does the rate of heat loss increase?

**P5.37** The volume flow $Q$ through an orifice plate is a function of pipe diameter $D$, pressure drop $\Delta p$ across the orifice, fluid density $\rho$ and viscosity $\mu$, and orifice diameter $d$. Using $D$, $\rho$, and $\Delta p$ as repeating variables, express this relationship in dimensionless form.

**P5.38** The size $d$ of droplets produced by a liquid spray nozzle is thought to depend on the nozzle diameter $D$, jet velocity $U$, and the properties of the liquid $\rho$, $\mu$, and $\Upsilon$. Rewrite this relation in dimensionless form. *Hint:* Take $D$, $\rho$, and $U$ as repeating variables.

**P5.39** The volume flow $Q$ over a certain dam is a function of dam width $b$, gravity $g$, and the upstream water depth $H$ above the dam crest. It is known that $Q$ is proportional to $b$. If $b = 120$ ft and $H = 15$ in., the flow rate is 600 ft$^3$/s. What will be the flow rate if $H = 3$ ft?

**P5.40** The time $t_d$ to drain a liquid from a hole in the bottom of a tank is a function of the hole diameter $d$, the initial fluid volume $\upsilon_0$, the initial liquid depth $h_0$, and the density $\rho$ and viscosity $\mu$ of the fluid. Rewrite this relation as a dimensionless function, using Ipsen's method.

**P5.41** A certain axial flow turbine has an output torque $M$ that is proportional to the volume flow rate $Q$ and also depends on the density $\rho$, rotor diameter $D$, and rotation rate $\Omega$. How does the torque change due to a doubling of (*a*) $D$ and (*b*) $\Omega$?

**P5.42** When disturbed, a floating buoy will bob up and down at frequency $f$. Assume that this frequency varies with buoy mass $m$, waterline diameter $d$, and the specific weight $\gamma$ of the liquid. (*a*) Express this as a dimensionless function. (*b*) If $d$ and $\gamma$ are constant and the buoy mass is halved, how will the frequency change?

## Nondimensionalizing the basic equations

**P5.43** Nondimensionalize the energy equation (4.75) and its boundary conditions (4.62), (4.63), and (4.70) by defining $T^* = T/T_0$, where $T_0$ is the inlet temperature, assumed constant. Use other dimensionless variables as needed from Eqs. (5.23). Isolate all dimensionless parameters you find, and relate them to the list given in Table 5.2.

**P5.44** The differential energy equation for incompressible two-dimensional flow through a "Darcy-type" porous medium is approximately

$$\rho c_p \frac{\sigma}{\mu}\frac{\partial p}{\partial x}\frac{\partial T}{\partial x} + \rho c_p \frac{\sigma}{\mu}\frac{\partial p}{\partial y}\frac{\partial T}{\partial y} + k\frac{\partial^2 T}{\partial y^2} = 0$$

where $\sigma$ is the *permeability* of the porous medium. All other symbols have their usual meanings. (*a*) What are the appropriate dimensions for $\sigma$? (*b*) Nondimensionalize this equation, using $(L, U, \rho, T_0)$ as scaling constants, and discuss any dimensionless parameters that arise.

**P5.45** A model differential equation, for chemical reaction dynamics in a plug reactor, is as follows:

$$u\frac{\partial C}{\partial x} = D\frac{\partial^2 C}{\partial x^2} - kC - \frac{\partial C}{\partial t}$$

where $u$ is the velocity, $D$ is a diffusion coefficient, $k$ is a reaction rate, $x$ is distance along the reactor, and $C$ is the (dimensionless) concentration of a given chemical in the reactor. (*a*) Determine the appropriate dimensions of $D$ and $k$. (*b*) Using a characteristic length scale $L$ and average velocity $V$ as parameters, rewrite this equation in dimensionless form and comment on any pi groups appearing.

**P5.46** If a vertical wall at temperature $T_w$ is surrounded by a fluid at temperature $T_0$, a natural convection boundary layer flow will form. For laminar flow, the momentum equation is

$$\rho(u\frac{\partial u}{\partial x} + \upsilon\frac{\partial u}{\partial y}) = \rho\beta(T - T_0)g + \mu\frac{\partial^2 u}{\partial y^2}$$

to be solved, along with continuity and energy, for $(u, v, T)$ with appropriate boundary conditions. The quantity $\beta$ is the thermal expansion coefficient of the fluid. Use $\rho$, $g$, $L$, and $(T_w - T_0)$ to nondimensionalize this equation. Note that there is no "stream" velocity in this type of flow.

**P5.47** The differential equation for small-amplitude vibrations $y(x, t)$ of a simple beam is given by

$$\rho A\frac{\partial^2 y}{\partial t^2} + EI\frac{\partial^4 y}{\partial x^4} = 0$$

where $\rho$ = beam material density
$A$ = cross-sectional area
$I$ = area moment of inertia
$E$ = Young's modulus

Use only the quantities $\rho$, $E$, and $A$ to nondimensionalize $y$, $x$, and $t$, and rewrite the differential equation in dimensionless form. Do any parameters remain? Could they be removed by further manipulation of the variables?

## Data for spheres, cylinders, other bodies

**P5.48** A smooth steel (SG = 7.86) sphere is immersed in a stream of ethanol at 20°C moving at 1.5 m/s. Estimate its drag in N from Fig. 5.3*a*. What stream velocity would quadruple its drag? Take $D = 2.5$ cm.

**P5.49** The sphere in Prob. P5.48 is dropped in gasoline at 20°C. Ignoring its acceleration phase, what will its terminal (constant) fall velocity be, from Fig. 5.3*a*?

**P5.50** The parachute in the chapter-opener photo is, of course, meant to decelerate the payload on Mars. The wind tunnel test gave a drag coefficient of about 1.1, based upon the projected area of the parachute. Suppose it was falling on *earth* and, at an altitude of 1000 m, showed a steady descent rate of about 18 mi/h. Estimate the weight of the payload.

**P5.51** A ship is towing a sonar array that approximates a submerged cylinder 1 ft in diameter and 30 ft long with its axis normal to the direction of tow. If the tow speed is 12 kn (1 kn = 1.69 ft/s), estimate the horsepower required to tow this cylinder. What will be the frequency of vortices shed from the cylinder? Use Figs. 5.2 and 5.3.

**P5.52** When fluid in a long pipe starts up from rest at a uniform acceleration $a$, the initial flow is laminar. The flow undergoes transition to turbulence at a time $t^*$ which depends, to first approximation, only upon $a$, $\rho$, and $\mu$. Experiments by

P. J. Lefebvre, on water at 20°C starting from rest with 1-g acceleration in a 3-cm-diameter pipe, showed transition at $t^* = 1.02$ s. Use this data to estimate (*a*) the transition time and (*b*) the transition Reynolds number $Re_D$ for water flow accelerating at 35 m/s$^2$ in a 5-cm-diameter pipe.

**P5.53** Vortex shedding can be used to design a *vortex flowmeter* (Fig. 6.34). A blunt rod stretched across the pipe sheds vortices whose frequency is read by the sensor downstream. Suppose the pipe diameter is 5 cm and the rod is a cylinder of diameter 8 mm. If the sensor reads 5400 counts per minute, estimate the volume flow rate of water in m$^3$/h. How might the meter react to other liquids?

**P5.54** A fishnet is made of 1-mm-diameter strings knotted into 2 × 2 cm squares. Estimate the horsepower required to tow 300 ft$^2$ of this netting at 3 kn in seawater at 20°C. The net plane is normal to the flow direction.

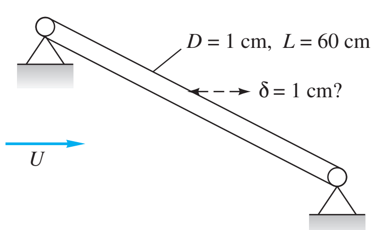
**P5.55** The radio antenna on a car begins to vibrate wildly at 8 Hz when the car is driven at 45 mi/h over a rutted road that approximates a sine wave of amplitude 2 cm and wavelength $\lambda = 2.5$ m. The antenna diameter is 4 mm. Is the vibration due to the road or to vortex shedding?

**P5.56** Flow past a long cylinder of square cross-section results in more drag than the comparable round cylinder. Here are data taken in a water tunnel for a square cylinder of side length $b = 2$ cm:

| $V$, m/s | 1.0 | 2.0 | 3.0 | 4.0 |
|---|---|---|---|---|
| Drag, N/(m of depth) | 21 | 85 | 191 | 335 |

(*a*) Use these data to predict the drag force per unit depth of wind blowing at 6 m/s, in air at 20°C, over a tall square chimney of side length $b = 55$ cm. (*b*) Is there any uncertainty in your estimate?

**P5.57** The simply supported 1040 carbon-steel rod of Fig. P5.57 is subjected to a crossflow stream of air at 20°C and 1 atm. For what stream velocity $U$ will the rod center deflection be approximately 1 cm?




**P5.57**

**P5.58** For the steel rod of Prob. P5.57, at what airstream velocity $U$ will the rod begin to vibrate laterally in resonance in its first mode (a half sine wave)? *Hint:* Consult a vibration text [34,35] under "lateral beam vibration."

**P5.59** A long, slender, smooth 3-cm-diameter flagpole bends alarmingly in 20 mi/h sea-level winds, causing patriotic citizens to gasp. An engineer claims that the pole will bend less if its surface is deliberately roughened. Is she correct, at least qualitatively?

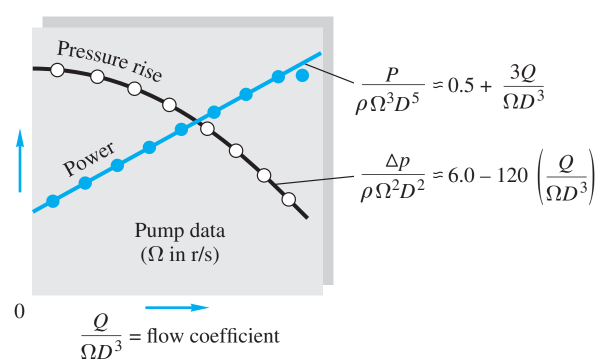
## Scaling of model data

***P5.60** The thrust $F$ of a free propeller, either aircraft or marine, depends upon density $\rho$, the rotation rate $n$ in r/s, the diameter $D$, and the forward velocity $V$. Viscous effects are slight and neglected here. Tests of a 25-cm-diameter model aircraft propeller, in a sea-level wind tunnel, yield the following thrust data at a velocity of 20 m/s:

| Rotation rate, r/min | 4800 | 6000 | 8000 |
|---|---|---|---|
| Measured thrust, N | 6.1 | 19 | 47 |

(*a*) Use this data to make a crude but effective dimensionless plot. (*b*) Use the dimensionless data to predict the thrust, in newtons, of a similar 1.6-m-diameter prototype propeller when rotating at 3800 r/min and flying at 225 mi/h at 4000-m standard altitude.

**P5.61** If viscosity is neglected, typical pump flow results from Example 5.3 are shown in Fig. P5.61 for a model pump tested in water. The pressure rise decreases and the power required increases with the dimensionless flow coefficient. Curve-fit expressions are given for the data. Suppose a similar pump of 12-cm diameter is built to move gasoline at 20°C and a flow rate of 25 m$^3$/h. If the pump rotation speed is 30 r/s, find (*a*) the pressure rise and (*b*) the power required.

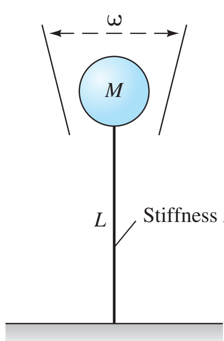



**P5.61**

**P5.62** For the system of Prob. P5.22, assume that a small model wind turbine of diameter 90 cm, rotating at 1200 r/min, delivers 280 watts when subjected to a wind of 12 m/s. The data is to be used for a prototype of diameter 50 m and winds of 8 m/s. For dynamic similarity, estimate (*a*) the rotation rate, and (*b*) the power delivered by the prototype. Assume sea-level air density.

***P5.63** The Keystone Pipeline in the Chapter 6 opener photo has $D = 36$ in. and an oil flow rate $Q = 590{,}000$ barrels per day (1 barrel = 42 U.S. gallons). Its pressure drop per unit length, $\Delta p/L$, depends on the fluid density $\rho$, viscosity $\mu$, diameter $D$, and flow rate $Q$. A water-flow model test, at 20°C, uses a 5-cm-diameter pipe and yields $\Delta p/L \approx 4000$ Pa/m. For dynamic similarity, estimate $\Delta p/L$ of the pipeline. For the oil take $\rho = 860$ kg/m$^3$ and $\mu = 0.005$ kg/m · s.

**P5.64** The natural frequency $\omega$ of vibration of a mass $M$ attached to a rod, as in Fig. P5.64, depends only on $M$




**P5.64**

and the stiffness $EI$ and length $L$ of the rod. Tests with a 2-kg mass attached to a 1040 carbon steel rod of diameter 12 mm and length 40 cm reveal a natural frequency of 0.9 Hz. Use these data to predict the natural frequency of a 1-kg mass attached to a 2024 aluminum alloy rod of the same size.

**P5.65** In turbulent flow near a flat wall, the local velocity $u$ varies only with distance $y$ from the wall, wall shear stress $\tau_w$, and fluid properties $\rho$ and $\mu$. The following data were taken in the University of Rhode Island wind tunnel for airflow, $\rho = 0.0023$ slug/ft$^3$, $\mu = 3.81$ E-7 slug/(ft · s), and $\tau_w = 0.029$ lbf/ft$^2$:

| $y$, in | 0.021 | 0.035 | 0.055 | 0.080 | 0.12 | 0.16 |
|---|---|---|---|---|---|---|
| $u$, ft/s | 50.6 | 54.2 | 57.6 | 59.7 | 63.5 | 65.9 |

(*a*) Plot these data in the form of dimensionless $u$ versus dimensionless $y$, and suggest a suitable power-law curve fit. (*b*) Suppose that the tunnel speed is increased until $u = 90$ ft/s at $y = 0.11$ in. Estimate the new wall shear stress, in lbf/ft$^2$.

**P5.66** A torpedo 8 m below the surface in 20°C seawater cavitates at a speed of 21 m/s when atmospheric pressure is 101 kPa. If Reynolds number and Froude number effects are negligible, at what speed will it cavitate when running at a depth of 20 m? At what depth should it be to avoid cavitation at 30 m/s?

**P5.67** A student needs to measure the drag on a prototype of characteristic dimension $d_p$ moving at velocity $U_p$ in air at standard atmospheric conditions. He constructs a model of characteristic dimension $d_m$, such that the ratio $d_p/d_m$ is some factor $f$. He then measures the drag on the model at dynamically similar conditions (also with air at standard atmospheric conditions). The student claims that the drag force on the prototype will be identical to that measured on the model. Is this claim correct? Explain.

**P5.68** For the rotating-cylinder function of Prob. P5.20, if $L >> D$, the problem can be reduced to only two groups, $F/(\rho U^2 LD)$ versus $(\Omega D/U)$. Here are experimental data for a cylinder 30 cm in diameter and 2 m long, rotating in sea-level air, with $U = 25$ m/s.

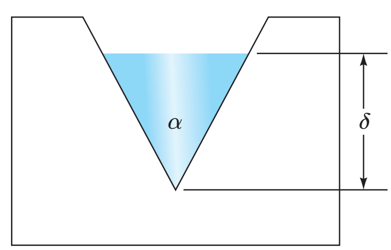
| $\Omega$, rev/min | 0 | 3000 | 6000 | 9000 | 12000 | 15000 |
|---|---|---|---|---|---|---|
| $F$, N | 0 | 850 | 2260 | 2900 | 3120 | 3300 |

(*a*) Reduce this data to the two dimensionless groups and make a plot. (*b*) Use this plot to predict the lift of a cylinder with $D = 5$ cm, $L = 80$ cm, rotating at 3800 rev/min in water at $U = 4$ m/s.

**P5.69** A simple flow measurement device for streams and channels is a notch, of angle $\alpha$, cut into the side of a dam, as shown in Fig. P5.69. The volume flow $Q$ depends only on $\alpha$, the acceleration of gravity $g$, and the height $\delta$ of the upstream water surface above the notch vertex. Tests of a model notch, of angle $\alpha = 55°$, yield the following flow rate data:

| $\delta$, cm | 10 | 20 | 30 | 40 |
|---|---|---|---|---|
| $Q$, m$^3$/h | 8 | 47 | 126 | 263 |

(*a*) Find a dimensionless correlation for the data. (*b*) Use the model data to predict the flow rate of a prototype notch, also of angle $\alpha = 55°$, when the upstream height $\delta$ is 3.2 m.




**P5.69**

**P5.70** A diamond-shaped body, of characteristic length 9 in, has the following measured drag forces when placed in a wind tunnel at sea-level standard conditions:

| $V$, ft/s | 30 | 38 | 48 | 56 | 61 |
|---|---|---|---|---|---|
| $F$, lbf | 1.25 | 1.95 | 3.02 | 4.05 | 4.81 |

Use these data to predict the drag force of a similar 15-in diamond placed at similar orientation in 20°C water flowing at 2.2 m/s.

**P5.71** The pressure drop in a venturi meter (Fig. P3.128) varies only with the fluid density, pipe approach velocity, and diameter ratio of the meter. A model venturi meter tested in water at 20°C shows a 5-kPa drop when the approach velocity is 4 m/s. A geometrically similar prototype meter is used to measure gasoline at 20°C and a flow rate of 9 $m^3$/min. If the prototype pressure gage is most accurate at 15 kPa, what should the upstream pipe diameter be?

**P5.72** A one-twelfth-scale model of a large commercial aircraft is tested in a wind tunnel at 20°C and 1 atm. The model chord length is 27 cm, and its wing area is 0.63 $m^2$. Test results for the drag of the model are as follows:

| V, mi/h | 50 | 75 | 100 | 125 |
|---|---|---|---|---|
| Drag, N | 15 | 32 | 53 | 80 |

In the spirit of Fig. 5.8, use this data to estimate the drag of the full-scale aircraft when flying at 550 mi/h, for the same angle of attack, at 32,800 ft standard altitude.

**P5.73** The power $P$ generated by a certain windmill design depends on its diameter $D$, the air density $\rho$, the wind velocity $V$, the rotation rate $\Omega$, and the number of blades $n$. (*a*) Write this relationship in dimensionless form. A model windmill, of diameter 50 cm, develops 2.7 kW at sea level when $V = 40$ m/s and when rotating at 4800 r/min. (*b*) What power will be developed by a geometrically and dynamically similar prototype, of diameter 5 m, in winds of 12 m/s at 2000 m standard altitude? (*c*) What is the appropriate rotation rate of the prototype?

**P5.74** A one-tenth-scale model of a supersonic wing tested at 700 m/s in air at 20°C and 1 atm shows a pitching moment of 0.25 kN · m. If Reynolds number effects are negligible, what will the pitching moment of the prototype wing be if it is flying at the same Mach number at 8-km standard altitude?

## Froude and Mach number scaling

**P5.75** According to the web site *USGS Daily Water Data for the Nation,* the mean flow rate in the New River near Hinton, WV, is 10,100 $ft^3$/s. If the hydraulic model in Fig. 5.9 is to match this condition with Froude number scaling, what is the proper model flow rate?

*__P5.76__ A 2-ft-long model of a ship is tested in a freshwater tow tank. The measured drag may be split into "friction" drag (Reynolds scaling) and "wave" drag (Froude scaling). The model data are as follows:

| Tow speed, ft/s | 0.8 | 1.6 | 2.4 | 3.2 | 4.0 | 4.8 |
|---|---|---|---|---|---|---|
| Friction drag, lbf | 0.016 | 0.057 | 0.122 | 0.208 | 0.315 | 0.441 |
| Wave drag, lbf | 0.002 | 0.021 | 0.083 | 0.253 | 0.509 | 0.697 |

The prototype ship is 150 ft long. Estimate its total drag when cruising at 15 kn in seawater at 20°C.

**P5.77** A dam 75 ft wide, with a nominal flow rate of 260 $ft^3$, is to be studied with a scale model 3 ft wide, using Froude scaling. (*a*) What is the expected flow rate for the model? (*b*) What is the danger of only using Froude scaling for this test? (*c*) Derive a formula for a force on the model as compared to a force on the prototype.

**P5.78** A prototype spillway has a characteristic velocity of 3 m/s and a characteristic length of 10 m. A small model is constructed by using Froude scaling. What is the minimum scale ratio of the model that will ensure that its minimum Weber number is 100? Both flows use water at 20°C.

**P5.79** An East Coast estuary has a tidal period of 12.42 h (the semidiurnal lunar tide) and tidal currents of approximately 80 cm/s. If a one-five-hundredth-scale model is constructed with tides driven by a pump and storage apparatus, what should the period of the model tides be and what model current speeds are expected?

**P5.80** A prototype ship is 35 m long and designed to cruise at 11 m/s (about 21 kn). Its drag is to be simulated by a 1-m-long model pulled in a tow tank. For Froude scaling find (*a*) the tow speed, (*b*) the ratio of prototype to model drag, and (*c*) the ratio of prototype to model power.

**P5.81** An airplane, of overall length 55 ft, is designed to fly at 680 m/s at 8000-m standard altitude. A one-thirtieth-scale model is to be tested in a pressurized helium wind tunnel at 20°C. What is the appropriate tunnel pressure in atm? Even at this (high) pressure, exact dynamic similarity is not achieved. Why?

**P5.82** A one-fiftieth-scale model of a military airplane is tested at 1020 m/s in a wind tunnel at sea-level conditions. The model wing area is 180 $cm^2$. The angle of attack is 3°. If the measured model lift is 860 N, what is the prototype lift, using Mach number scaling, when it flies at 10,000 m standard altitude under dynamically similar conditions? *Note:* Be careful with the area scaling.

**P5.83** A one-fortieth-scale model of a ship's propeller is tested in a tow tank at 1200 r/min and exhibits a power output of 1.4 ft · lbf/s. According to Froude scaling laws, what should the revolutions per minute and horsepower output of the prototype propeller be under dynamically similar conditions?

**P5.84** A prototype ocean platform piling is expected to encounter currents of 150 cm/s and waves of 12-s period and 3-m height. If a one-fifteenth-scale model is tested in a wave channel, what current speed, wave period, and wave height should be encountered by the model?

## Inventive rescaling of the data

*__P5.85__ As shown in Example 5.3, pump performance data can be nondimensionalized. Problem P5.61 gave typical dimensionless data for centrifugal pump "head," $H = \Delta p/\rho g$, as follows:

$$\frac{gH}{n^2D^2} \approx 6.0 - 120\left(\frac{Q}{nD^3}\right)^2$$

where $Q$ is the volume flow rate, $n$ the rotation rate in r/s, and $D$ the impeller diameter. This type of correlation allows one to compute $H$ when ($\rho$, $Q$, $D$) are known. (*a*) Show how to rearrange these pi groups so that one can *size* the pump, that is, compute $D$ directly when ($Q$, $H$, $n$) are known. (*b*) Make a crude but effective plot of your new function. (*c*) Apply part (*b*) to the following example: Find $D$ when $H = 37$ m, $Q = 0.14$ m$^3$/s, and $n = 35$ r/s. Find the pump diameter for this condition.

**P5.86** Solve Prob. P5.49 for glycerin at 20°C, using the modified sphere-drag plot of Fig. 5.11.

**P5.87** In Prob. P5.61 it would be difficult to solve for $\Omega$ because it appears in all three of the dimensionless pump coefficients. Suppose that, in Prob. 5.61, $\Omega$ is unknown but $D = 12$ cm and $Q = 25$ m$^3$/h. The fluid is gasoline at 20°C. Rescale the coefficients, using the data of Prob. P5.61, to make a plot of dimensionless power versus dimensionless rotation speed. Enter this plot to find the maximum rotation speed $\Omega$ for which the power will not exceed 300 W.

**P5.88** Modify Prob. P5.61 as follows: Let $\Omega = 32$ r/s and $Q = 24$ m$^3$/h for a geometrically similar pump. What is the maximum diameter if the power is not to exceed 340 W? Solve this problem by rescaling the data of Fig. P5.61 to make a plot of dimensionless power versus dimensionless diameter. Enter this plot directly to find the desired diameter.

**P5.89** Wall friction $\tau_w$, for turbulent flow at velocity $U$ in a pipe of diameter $D$, was correlated, in 1911, with a dimensionless correlation by Ludwig Prandtl's student H. Blasius:

$$\frac{\tau_w}{\rho U^2} \approx \frac{0.632}{(\rho UD/\mu)^{1/4}}$$

Suppose that ($\rho$, $U$, $\mu$, $\tau_w$) were all known and it was desired to find the unknown velocity $U$. Rearrange and rewrite the formula so that $U$ can be immediately calculated.

**P5.90** Knowing that $\Delta p$ is proportional to $L$, rescale the data of Example 5.10 to plot dimensionless $\Delta p$ versus dimensionless *viscosity*. Use this plot to find the viscosity required in the first row of data in Example 5.10 if the pressure drop is increased to 10 kPa for the same flow rate, length, and density.

***P5.91** The traditional "Moody-type" pipe friction correlation in Chap. 6 is of the form

$$f = \frac{2\Delta pD}{\rho V^2 L} = \text{fcn}\left(\frac{\rho VD}{\mu}, \frac{\varepsilon}{D}\right)$$

where $D$ is the pipe diameter, $L$ the pipe length, and $\varepsilon$ the wall roughness. Note that pipe average velocity $V$ is used on both sides. This form is meant to find $\Delta p$ when $V$ is known. (*a*) Suppose that $\Delta p$ is known, and we wish to find $V$. Rearrange the above function so that $V$ is isolated on the left-hand side. Use the following data, for $\varepsilon/D = 0.005$, to make a plot of your new function, with your velocity parameter as the ordinate of the plot.

| $f$ | 0.0356 | 0.0316 | 0.0308 | 0.0305 | 0.0304 |
|---|---|---|---|---|---|
| $\rho VD/\mu$ | 15,000 | 75,000 | 250,000 | 900,000 | 3,330,000 |

(*b*) Use your plot to determine $V$, in m/s, for the following pipe flow: $D = 5$ cm, $\varepsilon = 0.025$ cm, $L = 10$ m, for water flow at 20°C and 1 atm. The pressure drop $\Delta p$ is 110 kPa.

## Word Problems

**W5.1** In 98 percent of data analysis cases, the "reducing factor" $j$, which lowers the number $n$ of dimensional variables to $n - j$ dimensionless groups, exactly equals the number of relevant dimensions ($M$, $L$, $T$, $\Theta$). In one case (Example 5.5) this was not so. Explain in words why this situation happens.

**W5.2** Consider the following equation: 1 dollar bill ≈ 6 in. Is this relation dimensionally inconsistent? Does it satisfy the PDH? Why?

**W5.3** In making a dimensional analysis, what rules do you follow for choosing your scaling variables?

**W5.4** In an earlier edition, the writer asked the following question about Fig. 5.1: "Which of the three graphs is a more effective presentation?" Why was this a dumb question?

**W5.5** This chapter discusses the difficulty of scaling Mach and Reynolds numbers together (an airplane) and Froude and Reynolds numbers together (a ship). Give an example of a flow that would combine Mach and Froude numbers. Would there be scaling problems for common fluids?

**W5.6** What is different about a very *small* model of a weir or dam (Fig. P5.32) that would make the test results difficult to relate to the prototype?

**W5.7** What else are you studying this term? Give an example of a popular equation or formula from another course (thermodynamics, strength of materials, or the like) that does not satisfy the principle of dimensional homogeneity. Explain what is wrong and whether it can be modified to be homogeneous.

**W5.8** Some colleges (such as Colorado State University) have environmental wind tunnels that can be used to study phenomena like wind flow over city buildings. What details of scaling might be important in such studies?

**W5.9** If the model scale ratio is $\alpha = L_m/L_p$, as in Eq. (5.31), and the Weber number is important, how must the model and prototype surface tension be related to $\alpha$ for dynamic similarity?

**W5.10** For a typical incompressible velocity potential analysis in Chap. 8 we solve $\nabla^2\phi = 0$, subject to known values of $\partial\phi/\partial n$ on the boundaries. What dimensionless parameters govern this type of motion?

## Fundamentals of Engineering Exam Problems

**FE5.1** Given the parameters ($U$, $L$, $g$, $\rho$, $\mu$) that affect a certain liquid flow problem, the ratio $V^2/(Lg)$ is usually known as the
(*a*) velocity head, (*b*) Bernoulli head, (*c*) Froude number, (*d*) kinetic energy, (*e*) impact energy

**FE5.2** A ship 150 m long, designed to cruise at 18 kn, is to be tested in a tow tank with a model 3 m long. The appropriate tow velocity is
(*a*) 0.19 m/s, (*b*) 0.35 m/s, (*c*) 1.31 m/s, (*d*) 2.55 m/s, (*e*) 8.35 m/s

**FE5.3** A ship 150 m long, designed to cruise at 18 kn, is to be tested in a tow tank with a model 3 m long. If the model wave drag is 2.2 N, the estimated full-size ship wave drag is
(*a*) 5500 N, (*b*) 8700 N, (*c*) 38,900 N, (*d*) 61,800 N, (*e*) 275,000 N

**FE5.4** A tidal estuary is dominated by the semidiurnal lunar tide, with a period of 12.42 h. If a 1:500 model of the estuary is tested, what should be the model tidal period?
(*a*) 4.0 s, (*b*) 1.5 min, (*c*) 17 min, (*d*) 33 min, (*e*) 64 min

**FE5.5** A football, meant to be thrown at 60 mi/h in sea-level air ($\rho$ = 1.22 kg/m$^3$, $\mu$ = 1.78 E-5 N · s/m$^2$), is to be tested using a one-quarter scale model in a water tunnel ($\rho$ = 998 kg/m$^3$, $\mu$ = 0.0010 N · s/m$^2$). For dynamic similarity, what is the proper model water velocity?
(*a*) 7.5 mi/h, (*b*) 15.0 mi/h, (*c*) 15.6 mi/h, (*d*) 16.5 mi/h, (*e*) 30 mi/h

**FE5.6** A football, meant to be thrown at 60 mi/h in sea-level air ($\rho$ = 1.22 kg/m$^3$, $\mu$ = 1.78 E-5 N · m$^2$), is to be tested using a one-quarter scale model in a water tunnel ($\rho$ = 998 kg/m$^3$, $\mu$ = 0.0010 N · s/m$^2$). For dynamic similarity, what is the ratio of prototype force to model force?
(*a*) 3.86:1, (*b*) 16:1, (*c*) 32:1, (*d*) 56:1, (*e*) 64:1

**FE5.7** Consider liquid flow of density $\rho$, viscosity $\mu$, and velocity $U$ over a very small model spillway of length scale $L$, such that the liquid surface tension coefficient Y is important. The quantity $\rho U^2 L/Y$ in this case is important and is called the
(*a*) capillary rise, (*b*) Froude number, (*c*) Prandtl number, (*d*) Weber number, (*e*) Bond number

**FE5.8** If a stream flowing at velocity $U$ past a body of length $L$ causes a force $F$ on the body that depends only on $U$, $L$, and fluid viscosity $\mu$, then $F$ must be proportional to
(*a*) $\rho UL/\mu$, (*b*) $\rho U^2L^2$, (*c*) $\mu U/L$, (*d*) $\mu UL$, (*e*) $UL/\mu$

**FE5.9** In supersonic wind tunnel testing, if different gases are used, dynamic similarity requires that the model and prototype have the same Mach number and the same
(*a*) Euler number, (*b*) speed of sound, (*c*) stagnation enthalpy, (*d*) Froude number, (*e*) specific-heat ratio

**FE5.10** The Reynolds number for a 1-ft-diameter sphere moving at 2.3 mi/h through seawater (specific gravity 1.027, viscosity 1.07 E-3 N · s/m$^2$) is approximately
(*a*) 300, (*b*) 3000, (*c*) 30,000, (*d*) 300,000, (*e*) 3,000,000

**FE5.11** The Ekman number, important in physical oceanography, is a dimensionless combination of $\mu$, $L$, $\rho$, and the earth's rotation rate $\Omega$. If the Ekman number is proportional to $\Omega$, it should take the form
(*a*) $\rho\Omega^2L^2/\mu$, (*b*) $\mu\Omega L/\rho$, (*c*) $\rho\Omega L/\mu$, (*d*) $\rho\Omega L^2/\mu$, (*e*) $\rho\Omega/L\mu$

**FE5.12** A valid, but probably useless, dimensionless group is given by $(\mu T_0 g)/(\Upsilon L\alpha)$, where everything has its usual meaning, except $\alpha$. What are the dimensions of $\alpha$?
(*a*) $\Theta L^{-1}T^{-1}$, (*b*) $\Theta L^{-1}T^{-2}$, (*c*) $\Theta ML^{-1}$, (*d*) $\Theta^{-1}LT^{-1}$, (*e*) $\Theta LT^{-1}$

## Comprehensive Problems

**C5.1** Estimating pipe wall friction is one of the most common tasks in fluids engineering. For long circular rough pipes in turbulent flow, wall shear $\tau_w$ is a function of density $\rho$, viscosity $\mu$, average velocity $V$, pipe diameter $d$, and wall roughness height $e$. Thus, functionally, we can write $\tau_w$ = fcn($\rho$, $\mu$, $V$, $d$, $e$). (*a*) Using dimensional analysis, rewrite this function in dimensionless form. (*b*) A certain pipe has $d$ = 5 cm and $\varepsilon$ = 0.25 mm. For flow of water at 20°C, measurements show the following values of wall shear stress:

| $Q$, gal/min | 1.5 | 3.0 | 6.0 | 9.0 | 12.0 | 14.0 |
|---|---|---|---|---|---|---|
| $\tau_w$, Pa | 0.05 | 0.18 | 0.37 | 0.64 | 0.86 | 1.25 |

Plot these data using the dimensionless form obtained in part (*a*) and suggest a curve-fit formula. Does your plot reveal the entire functional relation obtained in part (*a*)?

**C5.2** When the fluid exiting a nozzle, as in Fig. P3.49, is a gas, instead of water, compressibility may be important, especially if upstream pressure $p_1$ is large and exit diameter $d_2$ is small. In this case, the difference $p_1 - p_2$ is no longer controlling, and the gas mass flow $\dot{m}$ reaches a maximum value that depends on $p_1$ and $d_2$ and also on the absolute upstream temperature $T_1$ and the gas constant $R$. Thus, functionally, $\dot{m} = \text{fcn}(p_1, d_2, T_1, R)$. (*a*) Using dimensional analysis, rewrite this function in dimensionless form. (*b*) A certain pipe has $d_2$ = 1 cm. For flow of air, measurements show the following values of mass flow through the nozzle:

| $T_1$, K | 300 | 300 | 300 | 500 | 800 |
|---|---|---|---|---|---|
| $p_1$, kPa | 200 | 250 | 300 | 300 | 300 |
| $\dot{m}$, kg/s | 0.037 | 0.046 | 0.055 | 0.043 | 0.034 |

Plot these data in the dimensionless form obtained in part (*a*). Does your plot reveal the entire functional relation obtained in part (*a*)?

**C5.3** Reconsider the fully developed draining vertical oil film problem (see Fig. P4.80) as an exercise in dimensional analysis. Let the vertical velocity be a function only of distance from the plate, fluid properties, gravity, and film thickness. That is, $w = \text{fcn}(x, \rho, \mu, g, \delta)$. (*a*) Use the pi theorem to rewrite this function in terms of dimensionless parameters. (*b*) Verify that the exact solution from Prob. P4.80 is consistent with your result in part (*a*).

**C5.4** The Taco Inc. model 4013 centrifugal pump has an impeller of diameter $D$ = 12.95 in. When pumping 20°C water at $\Omega$ = 1160 r/min, the measured flow rate $Q$ and pressure rise $\Delta p$ are given by the manufacturer as follows:

| $Q$, gal/min | 200 | 300 | 400 | 500 | 600 | 700 |
|---|---|---|---|---|---|---|
| $\Delta p$, lb/in$^2$ | 36 | 35 | 34 | 32 | 29 | 23 |

(*a*) Assuming that $\Delta p = \text{fcn}(\rho, Q, D, \Omega)$, use the pi theorem to rewrite this function in terms of dimensionless parameters and then plot the given data in dimensionless form. (*b*) It is desired to use the same pump, running at 900 r/min, to pump 20°C gasoline at 400 gal/min. According to your dimensionless correlation, what pressure rise $\Delta p$ is expected, in lbf/in$^2$?

**C5.5** Does an automobile radio antenna vibrate in resonance due to vortex shedding? Consider an antenna of length $L$ and diameter $D$. According to beam vibration theory [see [34] or [35, p. 401]], the first mode natural frequency of a solid circular cantilever beam is $\omega_n = 3.516[EI/(\rho A L^4)]^{1/2}$, where $E$ is the modulus of elasticity, $I$ is the area moment of inertia, $\rho$ is the beam material density, and $A$ is the beam cross-section area. (*a*) Show that $\omega_n$ is proportional to the antenna radius $R$. (*b*) If the antenna is steel, with $L$ = 60 cm and $D$ = 4 mm, estimate the natural vibration frequency, in Hz. (*c*) Compare with the shedding frequency if the car moves at 65 mi/h.

## Design Projects

**D5.1** We are given laboratory data, taken by Prof. Robert Kirchhoff and his students at the University of Massachusetts, for the spin rate of a 2-cup anemometer. The anemometer was made of ping-pong balls ($d$ = 1.5 in) split in half, facing in opposite directions, and glued to thin ($\frac{1}{4}$-in) rods pegged to a center axle. (See Fig. P7.91 for a sketch.) There were four rods, of lengths $l$ = 0.212, 0.322, 0.458, and 0.574 ft. The experimental data, for wind tunnel velocity $U$ and rotation rate $\Omega$, are as follows:

| $l$ = 0.212 | | $l$ = 0.322 | | $l$ = 0.458 | | $l$ = 0.574 | |
|---|---|---|---|---|---|---|---|
| **$U$, ft/s** | **$\Omega$, r/min** | **$U$, ft/s** | **$\Omega$, r/min** | **$U$, ft/s** | **$\Omega$, r/min** | **$U$, ft/s** | **$\Omega$, r/min** |
| 18.95 | 435 | 18.95 | 225 | 20.10 | 140 | 23.21 | 115 |
| 22.20 | 545 | 23.19 | 290 | 26.77 | 215 | 27.60 | 145 |
| 25.90 | 650 | 29.15 | 370 | 31.37 | 260 | 32.07 | 175 |
| 29.94 | 760 | 32.79 | 425 | 36.05 | 295 | 36.05 | 195 |
| 38.45 | 970 | 38.45 | 495 | 39.03 | 327 | 39.60 | 215 |

Assume that the angular velocity $\Omega$ of the device is a function of wind speed $U$, air density $\rho$ and viscosity $\mu$, rod length $l$, and cup diameter $d$. For all data, assume air is at 1 atm and 20°C. Define appropriate pi groups for this problem, and plot the data in this dimensionless manner. Comment on the possible uncertainty of the results.

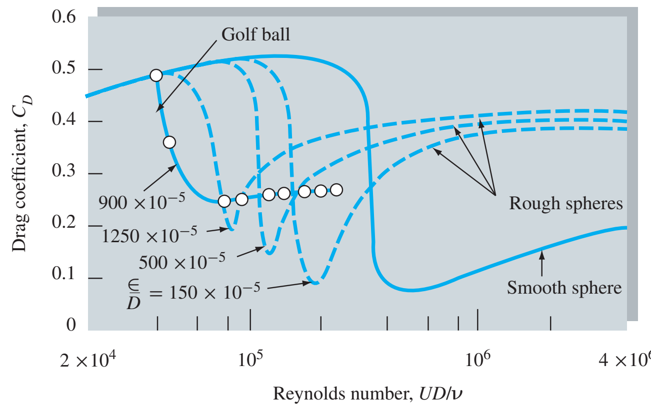
As a design application, suppose we are to use this anemometer geometry for a large-scale ($d$ = 30 cm) airport wind anemometer. If wind speeds vary up to 25 m/s and we desire an average rotation rate $\Omega$ = 120 r/min, what should be the proper rod length? What are possible limitations of your design? Predict the expected $\Omega$ (in r/min) of your design as affected by wind speeds from 0 to 25 m/s.

**D5.2** By analogy with the cylinder drag data in Fig. 5.3*b*, spheres also show a strong roughness effect on drag, at least in the Reynolds number range 4 E4 < $Re_D$ < 3 E5, which accounts for the dimpling of golf balls to increase their distance traveled. Some experimental data for roughened spheres [33] are given in Fig. D5.2. The figure also shows typical golf ball data. We see that some roughened spheres are better than golf balls in some regions. For the present study, let us neglect the ball's *spin,* which causes the very important side-force or *Magnus effect* (see Fig. 8.15) and assume that the ball is hit without spin and follows the equations of motion for plane motion ($x$, $z$):

$$m\dot{x} = -F\cos\theta \quad m\dot{z} = -F\sin\theta - W$$

where $$F = C_D\frac{\rho}{2}\frac{\pi}{4}D^2(\dot{x}^2 + \dot{z}^2) \quad \theta = \tan^{-1}\frac{\dot{z}}{\dot{x}}$$

The ball has a particular $C_D(\mathrm{Re}_D)$ curve from Fig. D5.2 and is struck with an initial velocity $V_0$ and angle $\theta_0$. Take the ball's average mass to be 46 g and its diameter to be 4.3 cm. Assuming sea-level air and a modest but finite range of initial conditions, integrate the equations of motion to compare the trajectory of "roughened spheres" to actual golf ball calculations. Can the rough sphere outdrive a normal golf ball for any conditions? What roughness-effect differences occur between a low-impact duffer and, say, Tiger Woods?




**D5.2**

## References


1. E. Buckingham, "On Physically Similar Systems: Illustrations of the Use of Dimensional Equations," *Phys. Rev.,* vol. 4, no. 4, 1914, pp. 345–376.
2. J. D. Anderson, *Computational Fluid Dynamics: The Basics with Applications,* McGraw-Hill, New York, 1995.
3. P. W. Bridgman, *Dimensional Analysis,* Yale University Press, New Haven, CT, 1922, rev. ed., 1963.
4. H. L. Langhaar, *Dimensional Analysis and the Theory of Models,* Wiley, New York, 1951.
5. E. C. Ipsen, *Units, Dimensions, and Dimensionless Numbers,* McGraw-Hill, New York, 1960.
6. H. G. Hornung, *Dimensional Analysis: Examples of the Use of Symmetry,* Dover, New York, 2006.
7. E. S. Taylor, *Dimensional Analysis for Engineers,* Clarendon Press, Oxford, England, 1974.
8. G. I. Barenblatt, *Dimensional Analysis,* Gordon and Breach, New York, 1987.
9. A. C. Palmer, *Dimensional Analysis and Intelligent Experimentation,* World Scientific Publishing, Hackensack, NJ, 2008.
10. T. Szirtes, *Applied Dimensional Analysis and Modeling*, 2d ed., Butterworth-Heinemann, Burlington, MA, 2006.
11. R. Esnault-Pelterie, *Dimensional Analysis and Metrology,* F. Rouge, Lausanne, Switzerland, 1950.
12. R. Kurth, *Dimensional Analysis and Group Theory in Astrophysics,* Pergamon, New York, 1972.
13. R. Kimball and M. Ross, *The Data Warehouse Toolkit: The Complete Guide to Dimensional Modeling*, 2d ed., Wiley, New York, 2002.
14. R. Nakon, *Chemical Problem Solving Using Dimensional Analysis,* Prentice-Hall, Upper Saddle River, NJ, 1990.
15. D. R. Maidment (ed.), *Hydrologic and Hydraulic Modeling Support: With Geographic Information Systems,* Environmental Systems Research Institute, Redlands, CA, 2000.
16. A. M. Curren, *Dimensional Analysis for Meds*, 4th ed., Delmar Cengage Learning, Independence, KY, 2009.
17. G. P. Craig, *Clinical Calculations Made Easy: Solving Problems Using Dimensional Analysis,* 4th ed., Lippincott Williams and Wilkins, Baltimore, MD, 2008.
18. M. Zlokarnik, *Dimensional Analysis and Scale-Up in Chemical Engineering,* Springer-Verlag, New York, 1991.
19. W. G. Jacoby, *Data Theory and Dimensional Analysis,* Sage, Newbury Park, CA, 1991.
20. B. Schepartz, *Dimensional Analysis in the Biomedical Sciences,* Thomas, Springfield, IL, 1980.
21. T. Horntvedt, *Calculating Dosages Safely: A Dimensional Analysis Approach*, F. A. Davis Co., Philadelphia, PA, 2012.
22. J. B. Bassingthwaighte et al., *Fractal Physiology,* Oxford Univ. Press, New York, 1994.
23. K. J. Niklas, *Plant Allometry: The Scaling of Form and Process,* Univ. of Chicago Press, Chicago, 1994.
24. "*Flow of Fluids through Valves, Fittings, and Pipes,*" Crane Valve Group, Long Beach, CA, 1957 (now updated as a CD-ROM; see <http://www.cranevalves.com>).
25. A. Roshko, "On the Development of Turbulent Wakes from Vortex Streets," *NACA Rep.* 1191, 1954.

26. G. W. Jones, Jr., "Unsteady Lift Forces Generated by Vortex Shedding about a Large, Stationary, Oscillating Cylinder at High Reynolds Numbers," *ASME Symp. Unsteady Flow,* 1968.
27. O. M. Griffin and S. E. Ramberg, "The Vortex Street Wakes of Vibrating Cylinders," *J. Fluid Mech.,* vol. 66, pt. 3, 1974, pp. 553–576.
28. *Encyclopedia of Science and Technology,* 11th ed., McGraw-Hill, New York, 2012.
29. J. Kunes, *Dimensionless Physical Quantities in Science and Engineering*, Elsevier, New York, 2012.
30. V. P. Singh et al. (eds.), *Hydraulic Modeling,* Water Resources Publications LLC, Highlands Ranch, CO, 1999.
31. L. Armstrong, *Hydraulic Modeling and GIS*, ESRI Press, La Vergne, TN, 2011.
32. R. Ettema, *Hydraulic Modeling: Concepts and Practice,* American Society of Civil Engineers, Reston, VA, 2000.
33. R. D. Blevins, *Applied Fluid Dynamics Handbook,* van Nostrand Reinhold, New York, 1984.
34. W. J. Palm III, *Mechanical Vibration*, Wiley, New York, 2006.
35. S. S. Rao, *Mechanical Vibrations,* 5th ed., Prentice-Hall, Upper Saddle River, NJ, 2010.
36. G. I. Barenblatt, *Scaling,* Cambridge University Press, Cambridge, UK, 2003.
37. L. J. Fingersh, "Unsteady Aerodynamics Experiment," *Journal of Solar Energy Engineering,* vol. 123, Nov. 2001, p. 267.
38. J. B. Barlow, W. H. Rae, and A. Pope, *Low-Speed Wind Tunnel Testing,* Wiley, New York, 1999.
39. B. H. Goethert, *Transonic Wind Tunnel Testing,* Dover, New York, 2007.
40. American Institute of Aeronautics and Astronautics, *Recommended Practice: Wind Tunnel Testing,* 2 vols., Reston, VA, 2003.
41. P. N. Desai, J. T. Schofield, and M. E. Lisano, "Flight Reconstruction of the Mars Pathfinder Disk-Gap-Band Parachute Drag Coefficients," *J. Spacecraft and Rockets*, vol. 42, no. 4, July–August 2005, pp. 672–676.
42. K.-H. Kim, "Recent Advances in Cavitation Research," 14th International Symposium on Transport Phenomena, Honolulu, HI, March 2012.