3 Uniform Particle Motion

The most common and perhaps the most important type of particle motion is steady, straight-line motion. This uniform motion is typically the result of the action of two forces, a constant external force such as gravity or electrical force and the resistance of the gas to particle motion. Analysis of uniform particle motion is especially useful for the study of aerosols, because in most situations aerosol particles come to a constant velocity almost instantly. The acceleration of aerosol particles is covered in Chapter 5. The resisting force of the gas depends on the relative velocity between the particle and the gas and is the same whether the particle moves through the gas or the gas flows past the particle. Both situations are used in the analysis of particle motion that follows.

3.1 NEWTON'S RESISTANCE LAW

The general equation for the force resisting the motion of a sphere passing through a gas was derived by Newton as part of a ballistic evaluation of cannonballs. Newton's resistance equation is valid for a wide range of particle motion but is useful primarily for Reynolds numbers greater than 1000, a range applicable to cannonballs, but not to aerosol particles. As is discussed in Section 3.2, Stokes's law, although derived from first principles, represents a special case of Newton's resistance law. Newton's resistance equation is used in Section 3.7 for high Reynolds number settling of aerosol particles.

Newton reasoned that the resistance experienced by a cannonball traveling through air is a result of the acceleration of the air that has to be pushed aside to allow the sphere to pass through. In one second, a sphere of diameter d will push aside a volume of gas equal to the projected area of the sphere times its velocity V. The mass of this amount of gas is

$$\dot{m} = \rho_g \frac{\pi}{4} d^2 V \tag{3.1}$$

The acceleration of the displaced gas is proportional to the relative velocity between the sphere and the gas, and thus the change of momentum per unit time is

$$\frac{\text{change of momentum}}{\text{unit time}} \propto \dot{m}V = \rho_g \frac{\pi}{4} d^2 V^2 \tag{3.2}$$

By definition, this rate of change of momentum is equal to the force required to move the sphere through the gas, the *gas resistance force* or *drag force* F_D, and given by

$$F_D = K\rho_g \frac{\pi}{4} d^2 V^2 \tag{3.3}$$

where K is a constant of proportionality. Newton originally thought that K was independent of velocity for a given shape. This is true, however, only when the Reynolds number for particle motion is greater than 1000. Equation 3.3 with a constant value of K is the restricted form of Newton's equation. A more general relationship is obtained by replacing K in Eq. 3.3 with the coefficient of drag C_D, giving

$$F_D = C_D \frac{\pi}{8} \rho_g d^2 V^2 \tag{3.4}$$

This is the general form of Newton's resistance equation, valid for all subsonic particle motion. The dimensionless coefficient of drag is constant for spheres having Re > 1000, but changes value for Re < 1000, as shown in Fig. 3.1. The curve in the figure is for spheres; other shapes have similar curves. For example, at Re > 1000, C_D for a delivery van is about 1.0, for a sedan about 0.5, for a sports car 0.2–0.3, and for an airplane 0.04.

The derivation of Eq. 3.3 is based on the inertia of the gas, and it follows that Eq. 3.3 is valid only for motion at high Reynolds numbers, where inertial forces are much larger than viscous forces. For particle motion with Reynolds numbers from 1000 to 2×10^5, C_D has a nearly constant value of 0.44, and Eq. 3.3 can be

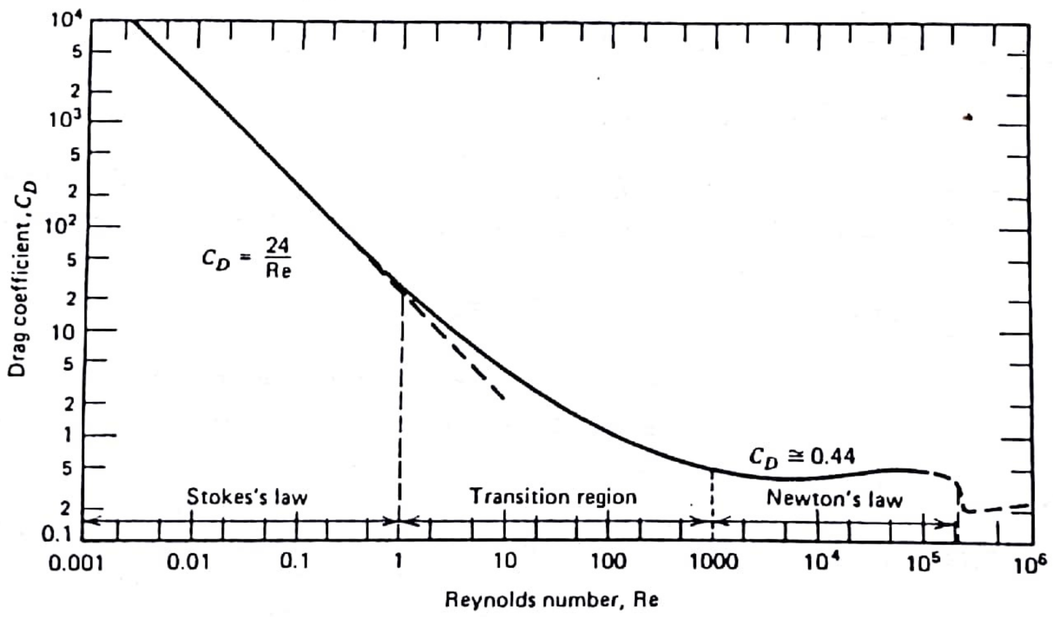

FIGURE 3.1 Drag coefficient versus Reynolds number for spheres.

used. With Reynolds numbers less than 1000, one needs to know the particle Reynolds number to get the correct value of C_D from Fig. 3.1 in order to use Eq. 3.4. Since most aerosol motion problems have either diameter or velocity as one of the unknowns, the Reynolds number cannot be calculated until the problem is solved. A way around this dilemma is covered in Section 3.7.

At the left side of Fig. 3.1, the straight-line segment on the log–log plot implies that the functional relationship between the variables is constant, but different from that given by Eq. 3.3. This is the Stokes region, which is covered in the next section. Between these straight-line regions, the relationship between C_D and Re can be described by

$$C_D = \frac{24}{\mathrm{Re}}(1 + 0.15\,\mathrm{Re}^{0.687}) \tag{3.5}$$

which agrees with the experimentally based correlation given by Clift et al. (1978) within 4% for Re < 800 and within 7% for Re < 1000.

3.2 STOKES'S LAW

Newton's drag (Re > 1000) applies to particle motion for which the viscous effects of the gas can be neglected compared with the inertial effects. In 1851 Stokes derived an expression for drag at the other extreme, when inertial forces are negligible compared with viscous forces. As discussed in Section 2.5, the Reynolds number is a ratio of inertial to viscous forces; consequently, a condition of negligible inertial forces, compared with viscous forces, implies a low Reynolds number and laminar flow. Because of the low velocities and small particle sizes involved, most aerosol motion occurs at low Reynolds numbers. Stokes's law thus has wide application to the study of aerosols, and it is worthwhile reviewing briefly the derivation of the law and its assumptions and implications.

Stokes's law is a solution of the generally unsolvable Navier–Stokes equations. These equations are the general differential equations describing fluid motion. They are derived from the application of Newton's second law to a fluid element on which the forces include body forces, pressure, and viscous forces. The resulting equations are very difficult to solve, because they are nonlinear partial differential equations. In general, some simplifying assumptions must be made before they can be solved. Stokes's solution does this by assuming that the inertial forces are negligibly small compared with the viscous forces. This assumption eliminates the higher order terms in the Navier–Stokes equations and yields linear equations that can be solved. Further assumptions of Stokes's derivation are that the fluid is incompressible, there are no walls or other particles nearby, the motion of the particle is constant, the particle is a rigid sphere, and the fluid velocity at the particle's surface is zero. A derivation of Stokes's law is given in the appendix at the end of this chapter.

Stokes solved the Navier–Stokes equations with the assumptions just stated to obtain equations for the forces acting at any point in the fluid surrounding a spherical

particle. The net force acting on the particle is obtained by integrating the normal and tangential forces over the surface of the particle. The two resulting forces acting in the direction opposite to particle motion are the *form component*,

$$F_n = \pi\eta V d \tag{3.6}$$

and the *frictional component*,

$$F_\tau = 2\pi\eta V d \tag{3.7}$$

These components are combined to give the total resisting force on a spherical particle moving with a velocity V through a fluid:

$$F_D = 3\pi\eta V d \tag{3.8}$$

This is *Stokes's law*. When the resisting force experienced by a particle is described by Eq. 3.8, the particles motion is said to be in the Stokes region.

As a particle moves through a fluid, it deforms the fluid, causing layers of the fluid in the region around the particle to slide over one another. The resisting force is the result of the friction of these layers sliding over one another. The energy spent overcoming this resistance is dissipated as heat throughout the fluid. In practice, the use of Stokes's law is restricted to situations in which the particle Reynolds number is less than 1.0. At this Reynolds number, the error in the drag force as calculated by Stokes's law is 12%; at a Reynolds number of 0.3, it is 5%.

It is instructive to compare the drag force given by Stokes's law with that given by Newton's law:

$$F_D = 3\pi\eta V d = C_D \frac{\pi}{8}\rho_g V^2 d^2 \quad \text{for Re} < 1 \tag{3.9}$$

Stokes's law contains viscosity, but not factors associated with inertia, such as ρ_g; Newton's law contains ρ_g, but not viscosity. Solving Eq. 3.9 for the coefficient of drag gives

$$C_D = \frac{24\eta}{\rho_g V d} = \frac{24}{\mathrm{Re}} \tag{3.10}$$

This is the equation for the straight-line portion at the left of Fig. 3.1. Because Eq. 3.10 includes V and d, it changes the functional relationship of Newton's equation from an equation having a drag force proportional to V^2 and d^2 to one with a drag force that is proportional to V and d, Stokes's law. For conditions between the range of application of Newton's and Stokes's laws, the functional dependence of drag gradually changes from V^2 to V and from d^2 to d. This is the curved portion of Fig. 3.1. The frictional component, Eq. 3.7, represents two-thirds of Stokes's drag, and the expression equivalent to Eq. 3.10 for flow in tubes where there is no form component gives $C_D = 16/\mathrm{Re}$, or two-thirds of Eq. 3.10.

How do the other assumptions of Stokes's law limit its application to aerosol particles? The assumption of an incompressible fluid implies not that air is incom-

pressible, but rather that it does not compress significantly near the particle as the particle moves through it. This is equivalent to assuming that the relative velocity is much less than the speed of sound, which is nearly always the case for aerosol particles.

The presence of a wall within 10 diameters of a particle will modify the drag force on the particle. Because of the small size of aerosol particles, only a tiny fraction of the aerosol particles will be within 10 particle diameters of a wall in any real container or tube.

The correction to Stokes's law for nonrigid spheres such as water droplets is generally insignificant. Water droplets settle 0.6% faster than predicted by Stokes's law because of circulations that develop within the droplet caused by the resisting force at the droplet surface.

The effect of nonzero velocity at the particle surface is covered in Section 3.4 and the effect of nonspherical particles in Section 3.5.

3.3 SETTLING VELOCITY AND MECHANICAL MOBILITY

One important application of Stokes's law is the determination of the velocity of an aerosol particle undergoing gravitational settling in still air. When a particle is released in air, it quickly reaches its terminal settling velocity, a condition of constant velocity wherein the drag force of the air on the particle, F_D, is exactly equal and opposite to the force of gravity F_G. Under this condition,

$$F_D = F_G = mg \tag{3.11}$$

$$3\pi\eta V d = \frac{(\rho_p - \rho_g)\pi d^3 g}{6} \tag{3.12}$$

where g is the acceleration of gravity, ρ_p is the density of the particle, and ρ_g is the density of the gas. The latter is included to account for the buoyancy effect, but this can usually be neglected because ρ_p is much greater than ρ_g. For example, a water droplet settling in air has a density ratio $\rho_p/\rho_g = 800$, and neglecting buoyancy introduces an error of only 0.1%. Solving Eq. 3.12 for the terminal settling velocity V_{TS} gives

$$V_{TS} = \frac{\rho_p d^2 g}{18\eta}, \quad \text{for } d > 1\ \mu\text{m and Re} < 1.0 \tag{3.13}$$

Terminal settling velocity increases rapidly with particle size, being proportional to the square of the particle diameter. As expected from the derivation, the settling velocity in the Stokes region is inversely proportional to viscosity and does not depend on the density of the gas. As will be shown in Section 5.2, aerosol particles adjust to their terminal settling velocity almost instantly, and V_{TS} is appropriate for characterizing particle motion in most real situations. Equation 3.13 cannot be used for particles smaller than 1.0 μm unless the slip correction factor, covered in the

next section, is applied. A simpler version of Eq. 3.13, valid only for spheres with standard density (1000 kg/m³ [1.0 g/cm³]) at standard conditions, is

$$\begin{cases} V_{TS} \cong 3 \times 10^{-5} d^2 \text{ m/s} & \text{for } 1 < d < 100\ \mu\text{m} \\ [V_{TS} \cong 0.003 d^2 \text{ cm/s} & \text{for } 1 < d < 100\ \mu\text{m}] \end{cases}$$

where d is in μm.

The terminal velocity for other kinds of external forces, such as centrifugal force, can be obtained by derivations similar to that given for gravity. In a centrifugal force field, the terminal velocity is

$$V_{TC} = \frac{\rho_p d^2 a_c}{18\eta} \tag{3.14}$$

where a_c is the centrifugal acceleration at the location of the particle. For a tangential velocity V_T and radius of motion R,

$$a_c = \frac{V_T^2}{R} \tag{3.15}$$

The Stokes settling equation, Eq. 3.13, is of fundamental importance to aerosol studies. However, in the form given, it is accurate (±10%) only for determining the settling velocity of standard-density particles having diameters of 1.5–75 μm. When slip correction is included (see Section 3.4), it is accurate for particles as small as 0.001 μm.

In Stokes's law, Eq. 3.8, the resistance force is directly proportional to velocity. From this relationship, we can define the particle mobility, B, a measure of the relative ease of producing steady motion for an aerosol particle, as

$$B = \frac{V}{F_D} = \frac{1}{3\pi\eta d} \quad \text{for } d > 1\ \mu\text{m} \tag{3.16}$$

Mobility is the ratio of the terminal velocity of a particle to the steady force producing that velocity. It has units of m/N · s [cm/dyn · s] and is often called mechanical mobility to distinguish it from electrical mobility. The terminal velocity of an aerosol particle is simply the force times mobility; for example,

$$V_{TS} = F_G B \tag{3.17}$$

For particles less than 1 μm, B must be multiplied by the slip correction factor. (See Section 3.4.)

EXAMPLE

What are the terminal settling velocity, drag force, and mobility of a 2.5-μm-diameter iron-oxide sphere settling in still air? The density of iron oxide is 5200 kg/m³ [5.2 g/cm³].

Substituting into Eq. 3.13, we obtain

$$V_{TS} = \frac{\rho_p d_p^2 g}{18\eta} = \frac{5200 \times (2.5 \times 10^{-6})^2 \times 9.81}{18(1.81 \times 10^{-5})} = 9.8 \times 10^{-4} \text{ m/s}$$

$$= \left[\frac{5.2 \times (2.5 \times 10^{-4})^2 \times 981}{18(1.81 \times 10^{-4})} = 0.098 \text{ cm/s}\right]$$

We calculate the particle Reynolds number to ensure that motion is in the Stokes region:

$$\text{Re} = 66{,}000 V d = 66{,}000(9.8 \times 10^{-4})(2.5 \times 10^{-6}) = 1.62 \times 10^{-4}$$

$$= [6.6 V d = 6.6(0.098)(2.5 \times 10^{-4}) = 1.62 \times 10^{-4}]$$

The motion is well within the Stokes region. The Stokes drag force is

$$F_D = 3\pi\eta V d = 3\pi(1.81 \times 10^{-5})(9.8 \times 10^{-4})(2.5 \times 10^{-6}) = 4.18 \times 10^{-11} \text{ N}$$

$$= [3\pi\eta V d = 3\pi(1.81 \times 10^{-4})(0.098)(2.5 \times 10^{-4}) = 4.18 \times 10^{-8} \text{ dyn}]$$

The same result can be obtained by calculating the force of gravity, since, at the terminal settling velocity, $F_D = F_G$.

$$F_G = ma = \rho_p \frac{\pi d^3}{6} g = \frac{5200\pi(2.5 \times 10^{-6})^3 9.81}{6} = 4.17 \times 10^{-13} \text{ N}$$

$$= \left[\frac{5.2\pi(2.5 \times 10^{-4})^3 981}{6} = 4.17 \times 10^{-8} \text{ dyn}\right]$$

The particle mobility is

$$B = \frac{V_{TS}}{F_G} = \frac{9.8 \times 10^{-4}}{4.17 \times 10^{-13}} = 2.35 \times 10^9 \text{ m/N} \cdot \text{s}$$

$$= \left[\frac{0.098}{4.17 \times 10^{-8}} = 2.35 \times 10^6 \text{ cm/s} \cdot \text{dyn}\right]$$

3.4 SLIP CORRECTION FACTOR

An important assumption of Stokes's law is that the relative velocity of the gas right at the surface of the sphere is zero. This assumption is not met for small particles

whose size approaches the mean free path of the gas. Such particles settle faster than predicted by Stokes's law, because there is "slip" at the surface of the particle. At standard conditions, this error becomes significant for particles less than 1 μm in diameter. In 1910, Cunningham derived a correction factor for Stokes's law to account for the effect of slip. The factor, called the *Cunningham correction factor* C_c, is always greater than one and reduces the Stokes drag force by

$$F_D = \frac{3\pi\eta V d}{C_c} \tag{3.18}$$

where

$$C_c = 1 + \frac{2.52\lambda}{d} \tag{3.19}$$

and λ is the mean free path, given by Eq. 2.25.

Use of the Cunningham correction factor, Eq. 3.19, extends the range of application of Stokes's law to particles of 0.1 μm in diameter. This range can be extended to still smaller particles by empirical equations based on experimental measurements of slip. The following equation agrees with the correlation (adjusted for mean free path) developed by Allen and Raabe for oil droplets (1982) *and* for solid particles (1985) within 2.1% for all particle sizes:

$$C_c = 1 + \frac{\lambda}{d}\left[2.34 + 1.05 \exp\left(-0.39 \frac{d}{\lambda}\right)\right] \tag{3.20}$$

This factor is called the *slip correction factor* and must be used in the form of Eq. 3.20 for particles less than 0.1 μm in diameter. The slip-corrected form of the terminal settling velocity becomes

$$V_{TS} = \frac{\rho_p d^2 g C_c}{18\eta} \quad \text{for Re} < 1.0 \tag{3.21}$$

Equation 3.21 is valid for all particle sizes when Re < 1.0 and Eq. 3.20 is used for C_c.

The slip correction factor for a 1.0-μm particle at standard conditions is 1.15; that is, the particle settles 15% faster than predicted by Eq. 3.13, which is based on the uncorrected form of Stokes's law. For particles less than 1 μm, slip increases rapidly as size decreases (see Figs. 3.2 and A12), and the Cunningham or slip correction factor must be used. For accurate work, it should be used for particles less than 5 or 10 μm. It is commonly stated that slip correction is necessary for particles as their size approaches the mean free path because "the particles are so small that they slip between the molecules." This is an incorrect, but useful, way of remembering how to apply the slip correction factor.

Slip correction increases as pressure decreases, because the mean free path increases. A useful form of Eq. 3.20 for air at pressures other than standard pressure is

$$C_c = 1 + \frac{1}{Pd}[15.60 + 7.00 \exp(-0.059Pd)] \quad (3.22)$$

where P is the absolute pressure in kPa and d is the particle diameter in μm.

Equation 3.22 underscores the dependence of slip correction on pressure and suggests a simple way to determine slip correction factors at pressures other than standard. Pressure appears in Eq. 3.22 only as a product of P and d, so any combination of P and d having the same product will give the same slip correction factor. Dividing Pd by 101 kPa gives the particle size that has the same slip correction factor at standard pressure as does a particle of size d at pressure P. This factor can be determined directly from Fig. 3.2 or Table A11. For example, the slip correction factor for a 1-μm particle at 202 kPa pressure is the same as that for a 2-μm particle at 101 kPa (C_c = 1.08), which can be obtained directly from Fig. 3.2 or Table A11.

Although pressure does not appear directly in Stokes's law, pressure affects the settling velocity of small particles because it influences the mean free path, which in turn affects slip correction. As shown in Table 3.1, increasing the gas density slows the settling of particles small enough to have a large slip correction (d < 0.1 μm) and particles large enough (d > 100 μm) to have a significant inertial component to their drag force. Particles between these sizes—for example, 10-μm particles—show little effect of pressure on settling velocity.

Motion of particles when $d << \lambda$ (d < 0.02 μm at standard pressure) is said to be in the free-molecule or molecular kinetic region. In this region the air resists the particle's motion, not as a continuous fluid, but as a series of discrete impacts occurring more frequently on the front of the particle than on the back. When d > 3 μm, particle motion is in the continuum region ($C_c \approx 1$), and the uncorrected form of Stokes's law, Eq. 3.8, can be used. The region between the molecular kinetic and the continuum region is called the transition region. The slip correction factor in the form of Eq. 3.20 corrects Stokes's law so that it can be used for particle motion in the free molecule and transition (as well as continuum) regions.

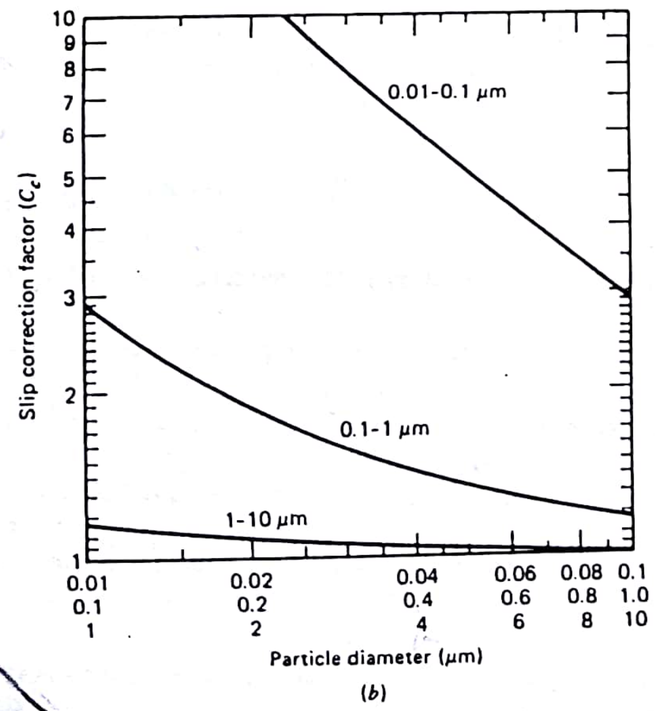

FIGURE 3.2 Slip correction factor at 20°C and 101 kPa.

TABLE 3.1 Effect of Pressure on Terminal Settling Velocity of Standard Density Spheres at 293 K [20°C].

Particle Diameter (μm)	V_{TS} at the Indicated Pressure (m/s)		
	P = 0.1 atm	P = 1.0 atm	P = 10 atm
0.001	6.9×10^{-8}	6.9×10^{-9}	6.9×10^{-10}
0.01	6.9×10^{-7}	7.0×10^{-8}	8.7×10^{-9}
0.1	7.0×10^{-6}	8.8×10^{-7}	3.5×10^{-7}
1	8.8×10^{-5}	3.5×10^{-5}	3.1×10^{-5}
10	0.0035	0.0031	0.0029
100	0.29	0.25	0.17

3.5 NONSPHERICAL PARTICLES

The equations for drag and settling velocity covered thus far are based on spherical particles. Liquid droplets (less than 1 mm) and some solid particles formed by condensation are spherical; most other types of particles are nonspherical. Some have regular geometric shapes, such as cubic (sea salt particles), cylindrical (bacteria and fibers), single crystals, or clusters of spheres. Others, such as agglomerated particles or crushed materials, have irregular shapes. The shape of a particle affects its drag force and settling velocity.

A correction factor called the *dynamic shape factor is* applied to Stokes's law to account for the effect of shape on particle motion. The dynamic shape factor is defined as the ratio of the actual resistance force of the nonspherical particle to the resistance force of a sphere having the same volume and velocity as the nonspherical particle. The dynamic shape factor χ is given by

$$\chi = \frac{F_D}{3\pi\eta V d_e} \quad (3.23)$$

where d_e, called the *equivalent volume diameter*, is the diameter of a sphere having the same volume as that of the irregular particle. This sphere is called the *equivalent volume sphere*. The equivalent volume diameter can be thought of as the diameter of the sphere that would result if the irregular particle melted to form a droplet. The calculation of d_e from microscopic measurements of particles is covered in Section 20.1. Stokes's law for irregular particles becomes

$$F_D = 3\pi\eta V d_e \chi \tag{3.24}$$

and the terminal settling velocity becomes

$$V_{TS} = \frac{\rho_p d_e^2 g}{18\eta\chi} \tag{3.25}$$

Dynamic shape factors for particles of various shapes are given in Table 3.2. Values for the geometric shapes were determined by measuring the settling velocity of geometric models in liquids. For irregular particles, settling velocities were measured indirectly using the elutriation devices described in Section 3.9. Values given in the table are averaged over all orientations, which is the usual situation for aerosol particle motion (Re < 0.1) because of the Brownian motion of the particles. Except for certain streamlined shapes, the dynamic shape factor is greater than 1.0. This means that nonspherical particles settle more slowly than their equivalent volume spheres.

The calculation of the slip correction factor for irregular particles is complicated, and the approximate factor calculated for the equivalent volume sphere is adequate

TABLE 3.2 Dynamic Shape Factors

	Dynamic Shape Factor,[a] χ			
		Axial Ratio		
Shape		2	5	10
Geometric Shapes				
Sphere	1.00			
Cube[b]	1.08			
Cylinder[b]				
Vertical axis		1.01	1.06	1.20
Horizontal axis		1.14	1.34	1.58
Orientation averaged		1.09	1.23	1.43
Straight chain[c]		1.10	1.35	1.68
Compact cluster				
Three spheres	1.15			
Four spheres	1.17			
Dusts				
Bituminous coal[d]	1.05–1.11			
Quartz[d]	1.36			
Sand[d]	1.57			
Talc[e]	1.88			

[a]Averaged over all orientations except where noted.
[b]Calculated from Johnson et al. (1987).
[c]Dahneke (1982).
[d]Davies (1979).
[e]Cheng et al. (1988).

for most irregular particles. The slip factor for randomly oriented fibers (L/d < 20) is 0–12% greater than that for the equivalent volume sphere. [See Dahneke (1973).]

Elongated particles such as fibers tend to line up with streamlines while flowing in tubes. These particles are oriented perpendicular to the direction of settling when settling occurs with a Reynolds number greater than about 10. For 0.1 < Re < 10 there is partial alignment, and below a Reynolds number of 0.1 there is no alignment.

3.6 AERODYNAMIC DIAMETER

An equivalent diameter that finds wide application in aerosol technology is the *aerodynamic diameter* d_a. This is defined, for a particular particle, as the diameter of the spherical particle with a density of 1000 kg/m³ [1 g/cm³] (the density of a water droplet) that has the same settling velocity as the particle. The aerodynamic diameter standardizes for shape (a sphere) and density (1000 kg/m³ [1 g/cm³]). A related, but less common, equivalent diameter is the *Stokes diameter* d_s, the diameter of the sphere that has the same density and settling velocity as the particle.

Equation 3.25 can be written in terms of these diameters, neglecting slip correction, as

$$V_{TS} = \frac{\rho_p d_e^2 g}{18\eta\chi} = \frac{\rho_0 d_a^2 g}{18\eta} = \frac{\rho_b d_s^2 g}{18\eta} \tag{3.26}$$

where ρ_0 is the standard particle density, 1000 kg/m³ [1.0 g/cm³]. The aerodynamic diameter can be thought of as the diameter of a water droplet having the same aerodynamic properties as the particle. If a particle has an aerodynamic diameter of 1 μm, it behaves in an aerodynamic sense like a 1-μm water droplet regardless of its shape, density, or physical size. Furthermore, it is aerodynamically indistinguishable from other particles of different size, shape, and density having aerodynamic diameters of 1 μm. The Stokes diameter is usually defined in terms of the normal density of the bulk material of the particle, ρ_b. This definition avoids the problem of defining the true density of the particle, which may be less than ρ_b due to porosity, occlusions, or an agglomerated structure.

An irregular particle and its aerodynamic and Stoke equivalent spheres are compared in Fig. 3.3. Each has the same settling velocity, but a different shape or density. Both aerodynamic and Stokes diameters are defined in terms of their aerodynamic behavior rather than their geometric properties. Settling velocity is a surrogate for most types of aerodynamic behavior. The aerodynamic diameter is the key particle property for characterizing filtration, respiratory deposition, and the performance of many types of air cleaners. In many situations, it is not necessary to know the true size, shape factor, and density of a particle if its aerodynamic diameter is known. Instruments such as elutriators (see Section 3.9) and cascade impactors (see Section 5.6) use aerodynamic separation to measure aerodynamic particle size.

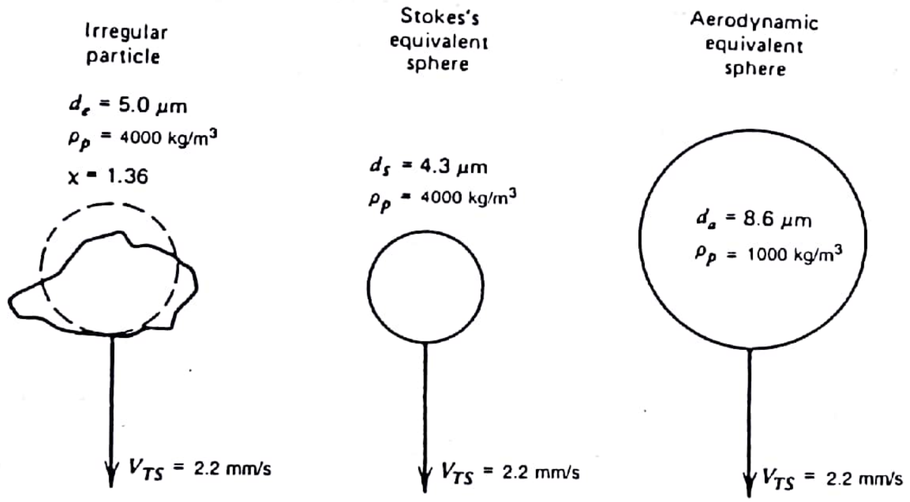

FIGURE 3.3 An irregular particle and its equivalent spheres.

Rearranging Eq. 3.26 gives

$$d_a = d_e\left(\frac{\rho_p}{\rho_0 \chi}\right)^{1/2} = d_s\left(\frac{\rho_b}{\rho_0}\right)^{1/2} \tag{3.27}$$

and, for spheres,

$$d_a = d_p\left(\frac{\rho_p}{\rho_0}\right)^{1/2} \tag{3.28}$$

For the usual case of a sphere with a density greater than 1000 kg/m³ [1.0 g/cm³], d_a is always greater than d_p.

For small particles for which slip correction must be applied, the foregoing definitions still hold, but are less convenient because the slip correction must be applied to the aerodynamic and Stokes diameters. The slip-corrected form of Eq. 3.28 is

$$d_a = d_p\left(\frac{C_c(d_p)}{C_c(d_a)}\right)^{1/2}\left(\frac{\rho_p}{\rho_0}\right)^{1/2}$$

where $C_c(d_p)$ and $C_c(d_a)$ are the respective slip correction factors for d_p and d_a. Likewise, when a particle's Reynolds numbers is greater than 1.0, its motion is outside the Stokes region, and although the preceding definitions hold, Eqs. 3.26–3.28 are not valid.

EXAMPLE

Calculate the aerodynamic diameter for a quartz particle with d_e = 20 μm and ρ_p = 2700 kg/m³.

$$d_a = d_e\left(\frac{\rho_p}{\rho_o \chi}\right)^{1/2} \quad \text{where } \chi = 1.36, \text{ from Table 3.2}$$

$$d_a = 20\left(\frac{2700}{1000 \times 1.36}\right)^{1/2} = 20(1.41) = 28.2\ \mu\text{m}$$

3.7 SETTLING AT HIGH REYNOLDS NUMBERS

For particle motion in the Stokes region, the settling velocity can be determined explicitly if the particle diameter and density are known; the diameter can be found if the velocity is known. For particle motion with Re > 1.0, this is not the case. Equating Eq. 3.4 to the force of gravity and solving for the settling velocity gives

$$V_{TS} = \left(\frac{4\rho_p d_p g}{3C_D \rho_g}\right)^{1/2} \tag{3.29}$$

In order to calculate V_{TS} by Eq. 3.29, we must determine the correct value of C_D. To obtain C_D from either Fig. 3.1 or Eq. 3.5, however, we need the particle Reynolds number ($\rho V d/\eta$), which requires knowing V_{TS}. This dilemma arises because the functional dependence of V_{TS} on d_p gradually changes from $V_{TS} \propto d^2$ in the Stokes region to $V_{TS} \propto d^{1/2}$ in the Newton's drag region. This changing relationship manifests itself in a changing value of C_D in Eq. 3.29 for the transition region, Reynolds number of 1–1000.

The first step is to determine whether the particle's motion is outside the Stokes region. By assuming that Stokes's law holds, Eq. 3.13 can be used to calculate V_{TS}, which then allows the calculation of the Reynolds number. If Re is greater than 1.0, the motion of the settling particle is outside the Stokes region and the calculated value of V_{TS} is wrong. In that case, the procedures outlined next can be used to determine the correct value of V_{TS}.

One way around the dilemma is an iterative (trial-and-error solution) obtained by substituting Eq. 3.5 into Eq. 3.29 and trying different values of velocity until Eq. 3.29 holds to within some desired accuracy. Table 3.3 gives V_{TS} for standard-density spheres at standard conditions computed using this process. The values in the table cannot be adjusted for different conditions because any change in ρ_p, ρ_g, or η will change the relationship between C_D and V_{TS}.

TABLE 3.3 Terminal Settling Velocity for 100- to 2000-Micrometer Spheres at Standard Conditions[a]

Diameter (μm)	Re	V_{TS} (m/s)	Diameter (μm)	Re	V_{TS} (m/s)
100	1.65	0.248	360	34.5	1.44
105	1.87	0.268	370	36.5	1.48
110	2.12	0.289	380	38.6	1.52
115	2.38	0.312	390	40.7	1.57
120	2.67	0.335	400	42.9	1.61
125	2.98	0.358	410	45.1	1.65
130	3.31	0.382	420	47.4	1.69
135	3.61	0.402	430	49.8	1.74
140	3.96	0.425	440	52.2	1.78
145	4.33	0.448	450	54.6	1.82
150	4.71	0.472	460	57.1	1.86
155	5.11	0.495	470	59.6	1.90
160	5.52	0.518	480	62.2	1.94
165	5.95	0.542	490	64.8	1.98
170	6.40	0.566	500	67.5	2.02
175	6.87	0.589	550	81.6	2.22
180	7.35	0.613	600	96.8	2.42
185	7.84	0.637	650	113	2.61
190	8.35	0.660	700	130	2.80
195	8.88	0.684	750	149	2.98
200	9.42	0.708	800	168	3.16
210	10.5	0.755	850	189	3.34
220	11.7	0.802	900	210	3.52
230	13.0	0.849	950	233	3.69
240	14.3	0.896	1000	257	3.86
250	15.7	0.943	1100	307	4.19
260	17.1	0.989	1200	361	4.52
270	18.6	1.03	1300	419	4.84
280	20.1	1.08	1400	480	5.15
290	21.7	1.12	1500	546	5.46
300	23.4	1.17	1600	614	5.77
310	25.1	1.21	1700	686	6.06
320	26.9	1.26	1800	762	6.36
330	28.7	1.30	1900	841	6.65
340	30.6	1.35	2000	924	6.94
350	32.5	1.39			

[a]Calculated using the intermediate-region approximation $C_D = (24/\mathrm{Re})(1 + 0.15\mathrm{Re}^{0.687})$. Particle density = 1000 kg/m³ [1.0 g/cm³].

Alternative approaches require the calculation of $C_D(\mathrm{Re})^2$. Rearranging Eq. 3.29 gives

$$C_D = \frac{4\rho_p d_p g}{3\rho_g V^2} \tag{3.30}$$

Multiplying both sides of Eq. 3.30 by $(\mathrm{Re})^2$ yields

$$C_D(\mathrm{Re})^2 = \frac{4d^3\rho_p\rho_g g}{3\eta^2} \tag{3.31}$$

The quantity $C_D(\mathrm{Re})^2$ can be determined because the right-hand side of Eq. 3.31 *does not include velocity*. $C_D(\mathrm{Re})^2$ can be used to determine the settling velocity *graphically* because it is a unique function of the Reynolds number that can be calculated without knowing the velocity. For a given particle Eq. 3.31 can be written

$$C_D(\mathrm{Re})^2 = K$$

$$C_D = \frac{K}{(\mathrm{Re})^2} \tag{3.32}$$

This is one equation relating C_D to Re that must be valid for the particle at its terminal settling velocity. The other is Eq. 3.5 for the transition region of Fig. 3.1. The two curves intersect at the correct value of Re and C_D. (See Fig. 3.4.) The settling velocity is determined from the Reynolds number of this point.

The settling velocity can be calculated directly using $C_D(\mathrm{Re})^2$ by the empirical equation

$$V_{TS} = \left(\frac{\eta}{\rho_g d_p}\right) \exp(-3.070 + 0.9935J - 0.0178J^2) \tag{3.33}$$

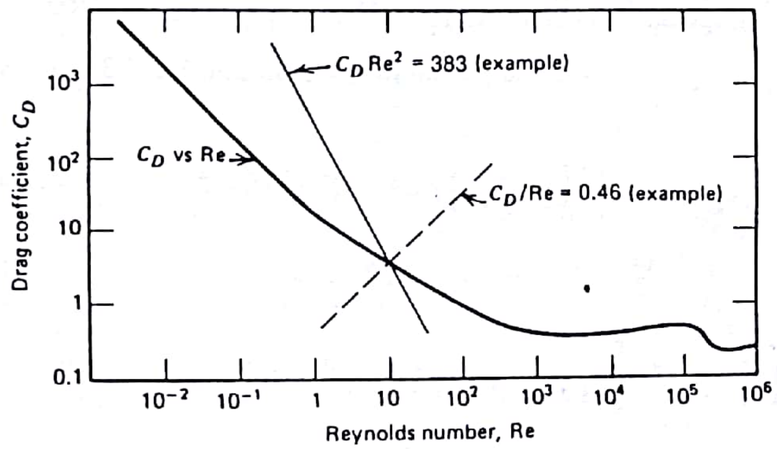

FIGURE 3.4 Drag coefficient versus Reynolds number, showing example lines for $C_D(\mathrm{Re})^2 = 383$ and $C_D/\mathrm{Re} = 0.46$.

where

$$J = \ln[C_D(\mathrm{Re})^2] = \ln\left(\frac{4\rho_p \rho_g d^3 g}{3\eta^2}\right)$$

Equation (3.33) agrees with iterative calculations using the transition region approximation, Eq. 3.5, within 3% for $1 < \mathrm{Re} < 600$ and within 7% for $0.5 < \mathrm{Re} < 1000$. If the Reynold number is less than 0.5, it is more accurate to use Eq. 3.13 or 3.21.

A slightly faster procedure is to use the values of Re tabulated for their corresponding values of $C_D(\mathrm{Re})^2$ given in Table 3.4. The value of $C_D(\mathrm{Re})^2$ is calculated first, and the value of Re associated with this calculated value is obtained from the table. V_{TS} is then calculated from this Reynolds number.

EXAMPLE

What is the gravitational settling velocity at standard conditions of a 100-μm-diameter sphere having a density of 8000 kg/m^3?

$$C_D(\mathrm{Re})^2 = \frac{4d^3\rho_p\rho_g g}{3\eta^2} = \frac{4 \times (100 \times 10^{-6})^3 \times 8000 \times 1.2 \times 9.81}{3 \times (1.81 \times 10^{-5})^2} = 383$$

$$\left[C_D(\mathrm{Re})^2 = \frac{4 \times (10^{-2})^3 \times 8 \times 1.20 \times 10^{-3} \times 981}{3(1.81 \times 10^{-4})^2} = 383\right]$$

The Reynolds number obtained by interpolation from Table 3.4 for a value of $C_D(\mathrm{Re})^2$ equal to 383 is 9.40, giving

(1) $$V_{TS} = \frac{\mathrm{Re}}{66{,}000d} = \frac{9.40}{66{,}000 \times 100 \times 10^{-6}} = 1.42 \text{ m/s} \quad \left[\frac{9.40}{6.6 \times 100 \times 10^{-4}} = 142 \text{ cm/s}\right]$$

Alternatively, substituting $J = \ln(383) = 5.95$ into Eq. 3.33 yields

(2) $$V_{TS} = \left(\frac{1.81 \times 10^{-5}}{1.20 \times 10^{-4}}\right) \exp[-3.07 + 0.9935(5.95) - 0.0178(5.95)^2]$$

$$= 0.151 \exp(2.21) = 1.38 \text{ m/s}$$

$$= \left[\left(\frac{1.81 \times 10^{-4}}{1.20 \times 10^{-3} \times 10^{-2}}\right) \exp(2.21) = 138 \text{ cm/s}\right]$$

In this example, if Stokes's law had been unwittingly applied, the calculated velocity would have been

TABLE 3.4 Reynolds Numbers for Settling Spheres Having $C_D(\mathrm{Re})^2$ from 10 to 10^5 [a]

$C_D(\mathrm{Re})^2$	0	1	2	3	4	5	6	7	8	9
10	0.40	0.43	0.47	0.51	0.54	0.58	0.61	0.65	0.69	0.72
20	0.75	0.79	0.82	0.86	0.89	0.92	0.96	0.99	1.02	1.06
30	1.09	1.12	1.15	1.18	1.22	1.25	1.28	1.31	1.34	1.37
40	1.40	1.43	1.47	1.50	1.53	1.56	1.59	1.62	1.65	1.68
50	1.71	1.74	1.77	1.80	1.83	1.86	1.89	1.92	1.95	1.98
60	2.01	2.04	2.07	2.10	2.13	2.16	2.19	2.22	2.25	2.27
70	2.30	2.33	2.36	2.39	2.42	2.45	2.47	2.50	2.53	2.56
80	2.59	2.62	2.64	2.67	2.70	2.73	2.75	2.78	2.81	2.84
90	2.86	2.89	2.92	2.95	2.97	3.00	3.03	3.05	3.08	3.11
$C_D(\mathrm{Re})^2$	0	10	20	30	40	50	60	70	80	90
100	3.14	3.40	3.66	3.92	4.17	4.41	4.66	4.90	5.13	5.36
200	5.59	5.82	6.05	6.27	6.49	6.70	6.92	7.13	7.34	7.55
300	7.75	7.96	8.16	8.36	8.56	8.75	8.95	9.14	9.34	9.53
400	9.72	9.90	10.1	10.3	10.5	10.6	10.8	11.0	11.2	11.4
500	11.5	11.7	11.9	12.1	12.2	12.4	12.6	12.8	12.9	13.1
600	13.3	13.4	13.6	13.8	13.9	14.1	14.2	14.4	14.6	14.7
700	14.9	15.0	15.2	15.4	15.5	15.7	15.8	16.0	16.1	16.3
800	16.4	16.6	16.7	16.9	17.1	17.2	17.3	17.5	17.6	17.8
900	17.9	18.1	18.2	18.4	18.5	18.7	18.8	19.0	19.1	19.2
$C_D(\mathrm{Re})^2$	0	100	200	300	400	500	600	700	800	900
1000	19.4	20.8	22.1	23.4	24.7	26.0	27.2	28.4	29.6	30.7
2000	31.8	33.0	34.0	35.1	36.2	37.2	38.2	39.2	40.2	41.2
3000	42.2	43.2	44.1	45.1	46.0	46.9	47.8	48.7	49.6	50.5
4000	51.4	52.2	53.1	53.9	54.8	55.6	56.4	57.3	58.1	58.9
5000	59.7	60.5	61.3	62.1	62.9	63.6	64.4	65.2	65.9	66.7
6000	67.4	68.2	68.9	69.7	70.4	71.1	71.8	72.6	73.3	74.0
7000	74.7	75.4	76.1	76.8	77.5	78.2	78.9	79.6	80.2	80.9
8000	81.6	82.3	82.9	83.6	84.2	84.9	85.6	86.2	86.9	87.5
9000	88.1	88.8	89.4	90.1	90.7	91.3	92.0	92.6	93.2	93.8
$C_D(\mathrm{Re})^2$	0	1000	2000	3000	4000	5000	6000	7000	8000	9000
10000	94.4	100	106	112	117	123	128	133	138	143
20000	148	152	157	162	166	170	175	179	183	187
30000	191	195	199	203	207	211	214	218	222	226
40000	229	233	236	240	243	247	250	254	257	260
50000	264	267	270	273	277	280	283	286	289	292
60000	295	298	301	304	307	310	313	316	319	322
70000	325	328	331	334	336	339	342	345	348	350
80000	353	356	358	361	364	367	369	372	374	377
90000	380	382	385	388	390	393	395	398	400	403

[a] The value of $C_D(\mathrm{Re})^2$ equals the sum of the row and column headings.

$$V_{TS} = \frac{\rho_p d^2 g}{18\eta} = \frac{8000 \times (10^{-4})^2 \times 9.81}{18 \times 1.81 \times 10^{-5}} = 2.41 \text{ m/s}$$

an overestimate of 70%.

The problem of determining a particle's diameter when its settling velocity is known is attacked in a similar way. The quantity C_D/Re, obtained by combining Eqs. 3.30 and 2.41, can be calculated without knowing the particle diameter:

$$\frac{C_D}{\text{Re}} = \frac{4\rho_p \eta g}{3\rho_g^2 V^3} \tag{3.34}$$

Equation 3.34 defines another relationship between C_D and Re that must be met for a settling particle. The particle diameter can be obtained by iterative, graphical or tabular procedures analogous to those for the settling velocity just described. Values of Reynolds numbers for C_D/Re from 0.01 to 100 are given in Table 3.5. The diameter of a settling particle whose velocity is known can be calculated directly using C_D/Re by the empirical equation.

$$d = \left(\frac{\eta}{\rho_g V_{TS}}\right) \exp(1.787 - 0.577H + 0.0109H^2) \tag{3.35}$$

where

$$H = \ln(C_D/\text{Re}) = \ln\left(\frac{4\rho_p \eta g}{3\rho_g^2 V^3}\right)$$

Equation 3.35 agrees with iterative calculations using the transition region approximation, Eq. 3.5, within 3% for $0.5 < \text{Re} < 1000$.

EXAMPLE

To work the preceding example backwards, assume that the particle diameter is not known, but its settling velocity at standard conditions has been determined to be 1.42 m/s [142 cm/s] and $\rho_p = 8000$ kg/m^3 [8 g/cm^3]. Then

$$\frac{C_D}{\text{Re}} = \frac{4\rho_p \eta g}{3\rho_g^2 V^3} = \frac{4 \times 8000 \times 1.81 \times 10^{-5} \times 9.81}{3(1.20)^2(1.42)^3} = 0.46$$

$$\left[\frac{C_D}{\text{Re}} = \frac{4 \times 8 \times 1.81 \times 10^{-4} \times 981}{3(1.2 \times 10^{-3})^2(142)^3} = 0.46\right]$$

TABLE 3.5 Reynolds Numbers for Settling Spheres Having C_D/Re from 0.01 to 100[a]

C_D/Re	0	0.001	0.002	0.003	0.004	0.005	0.006	0.007	0.008	0.009
0.01	106	99.5	93.8	88.8	84.4	80.6	77.2	74.1	71.3	68.8
0.02	66.5	64.3	62.4	60.6	58.9	57.3	55.8	54.5	53.2	52.0
0.03	50.8	49.7	48.7	47.8	46.8	46.0	45.1	44.3	43.6	42.8
0.04	42.1	41.5	40.8	40.2	39.6	39.1	38.5	38.0	37.5	37.0
0.05	36.5	36.0	35.6	35.2	34.7	34.3	34.0	33.6	33.2	32.8
0.06	32.5	32.2	31.8	31.5	31.2	30.9	30.6	30.3	30.0	29.7
0.07	29.5	29.2	29.0	28.7	28.5	28.2	28.0	27.8	27.5	27.3
0.08	27.1	26.9	26.7	26.5	26.3	26.1	25.9	25.7	25.5	25.3
0.09	25.2	25.0	24.8	24.7	24.5	24.3	24.2	24.0	23.9	23.7
C_D/Re	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.1	23.6	22.2	21.1	20.0	19.2	18.4	17.7	17.0	16.4	15.9
0.2	15.4	15.0	14.6	14.2	13.8	13.5	13.2	12.9	12.6	12.4
0.3	12.1	11.9	11.7	11.4	11.2	11.1	10.9	10.7	10.5	10.4
0.4	10.2	10.1	9.93	9.80	9.67	9.54	9.42	9.30	9.19	9.08
0.5	8.97	8.87	8.77	8.67	8.58	8.49	8.40	8.32	8.23	8.15
0.6	8.07	8.00	7.92	7.85	7.78	7.71	7.64	7.58	7.51	7.45
0.7	7.39	7.33	7.27	7.21	7.16	7.10	7.05	7.00	6.95	6.90
0.8	6.85	6.8	6.75	6.70	6.66	6.61	6.57	6.53	6.48	6.44
0.9	6.40	6.36	6.32	6.28	6.24	6.20	6.16	6.12	6.09	6.06
C_D/Re	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	6.03	5.71	5.44	5.20	4.99	4.8	4.63	4.48	4.34	4.21
2	4.09	3.98	3.88	3.79	3.70	3.62	3.54	3.47	3.40	3.33
3	3.27	3.21	3.16	3.11	3.06	3.01	2.96	2.92	2.88	2.84
4	2.80	2.76	2.72	2.69	2.66	2.62	2.59	2.56	2.53	2.51
5	2.48	2.45	2.43	2.40	2.38	2.35	2.33	2.31	2.29	2.27
6	2.25	2.23	2.21	2.19	2.17	2.15	113	2.12	2.10	2.08
7	2.07	2.05	2.04	2.02	2.01	2.00	1.98	1.97	1.96	1.94
8	1.93	1.92	1.90	1.89	1.88	1.87	1.86	1.84	1.83	1.82
9	1.81	1.80	1.79	1.78	1.77	1.76	1.75	1.74	1.73	1.72
C_D/Re	0	1	2	3	4	5	6	7	8	9
10	1.71	1.63	1.55	1.49	1.43	1.38	1.33	1.29	1.25	1.21
20	1.18	1.15	1.12	1.10	1.07	1.05	1.03	1.01	0.99	0.97
30	0.95	0.93	0.92	0.90	0.89	0.88	0.86	0.85	0.84	0.83
40	0.82	0.81	0.80	0.79	0.78	0.77	0.76	0.75	0.74	0.73
50	0.73	0.72	0.71	0.70	0.70	0.69	0.68	0.68	0.67	0.67
60	0.66	0.65	0.65	0.64	0.64	0.63	0.63	0.62	0.62	0.61
70	0.61	0.61	0.60	0.60	0.59	0.59	0.58	0.58	0.58	0.57
80	0.57	0.57	0.56	0.56	0.55	0.55	0.55	0.54	0.54	0.54
90	0.54	0.53	0.53	0.53	0.52	0.52	0.52	0.51	0.51	0.51

[a]The value of C_D/Re equals the sum of the row and column headings.

The Reynolds number obtained from Table 3.5 for $C_D/\mathrm{Re} = 0.46$ is 9.42, which allows calculation of particle diameter,

$$d = \frac{9.42}{66000 \times V_{TS}} = \frac{9.42}{66000 \times 1.42} = 101\ \mu\mathrm{m} \quad \left[\frac{9.42}{6.6 \times 142} = 101\ \mu\mathrm{m}\right]$$

Alternatively, we can use Eq. 3.35, substituting $H = \ln(0.46) = -0.777$ to obtain

$$d = \left(\frac{1.81 \times 10^{-5}}{1.2 \times 1.38}\right) \exp[1.787 - 0.577(-0.777) + 0.0109(-0.777)^2]$$

$$d = (1.06 \times 10^{-5}) \exp(2.242) = 100\ \mu\mathrm{m}$$

$$\left[d = \left(\frac{1.81 \times 10^{-4}}{1.20 \times 10^{-3} \times 142}\right) \exp(2.242) = 100\ \mu\mathrm{m}\right]$$

the same as the original diameter.

For nonspherical particles with motion outside the Stokes region, the right-hand sides of Eqs. 3.31 and 3.34 are multiplied by $1/\chi$, where χ is the dynamic shape factor, and the equivalent volume diameter is used throughout. The method for determining the velocity or diameter is the same as that given for spheres. This approach is valid for compact particles with Reynolds numbers less than 100 or for irregular particles with sharp edges and Reynolds numbers less than 10.

The calculations of settling velocity for spheres can be summarized as follows. For the smallest particles, <0.1 μm, use Eq. 3.21 with Eq. 3.20 for slip correction. For particles from 0.1 to 1 μm, use either Eq. 3.19 or 3.20 to calculate the slip correction factor. For particles >1.0 μm, Eq. 3.13 can be used, provided that the particle Reynolds number is less than 1.0. For particle motion with Reynolds number from 0.5 to 1.0, Eq. 3.13, Eq. 3.33, or the tabular procedure can be used. For large and heavy particles for which Re >1.0, we must use Eq. 3.33 or the tabular procedure using Eqs. 3.31 and 2.41 and Table 3.4. Equation 3.29 with $C_D = 0.44$ can be used for motion with $1000 < \mathrm{Re} < 2 \times 10^5$, although this is beyond the range of usual aerosol particle motion.

3.8 STIRRED SETTLING

The equations for V_{TS} permit us to evaluate how aerosol concentration will change with time in a room or a container. Unfortunately, the real case is likely to be very complicated, so we consider here the two idealized situations shown in Fig. 3.5; tranquil and stirred settling of monodisperse aerosols. For monodisperse aerosols, the real case lies between these two idealized situations, usually closer to the stirred settling case than to the tranquil settling case.

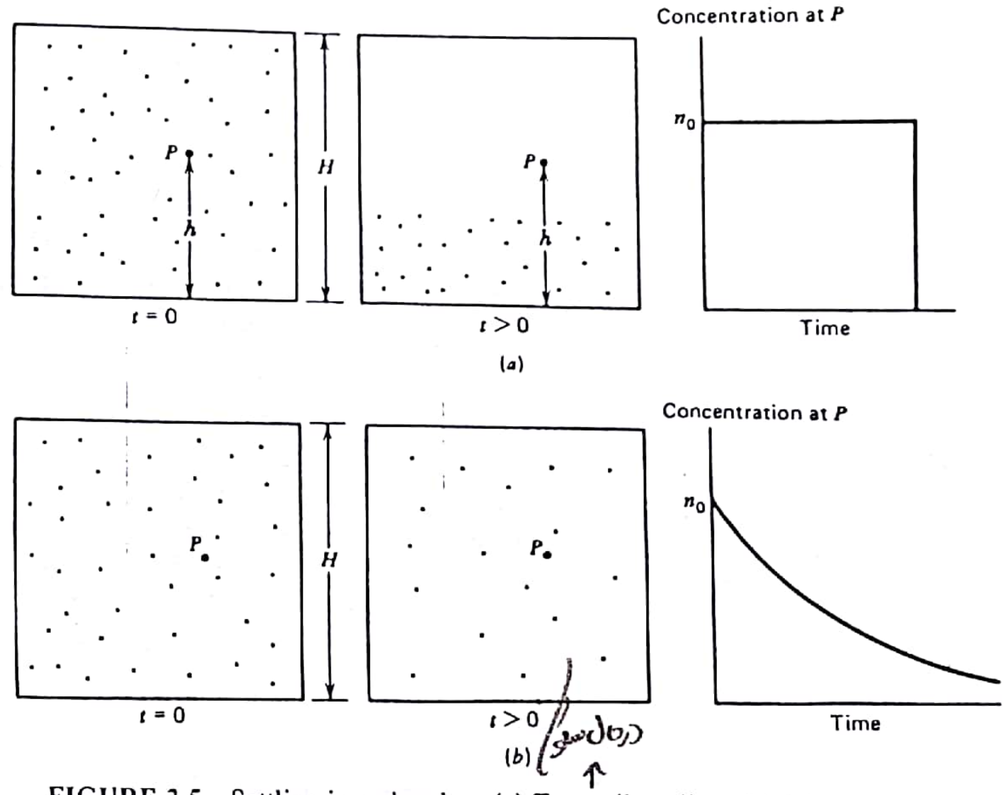

FIGURE 3.5 Settling in a chamber. (a) Tranquil settling. (b) Stirred settling.

In tranquil settling, there is no motion of the gas, and each particle's motion is solely its gravitational settling in the stagnant air mass. Consider a cylindrical chamber of height H containing at time $t = 0$ a number concentration n_0 of uniformly distributed monodisperse particles. If we neglect diffusion, all the particles will be settling with the same constant velocity V_{TS}. After a time t, there will be an upper region with no particles and a lower region containing an aerosol with a flat upper boundary having the original number concentration. (See Fig. 3.5a.) Some fraction of the particles will have deposited on the floor of the chamber. At any point within the chamber the concentration will remain constant, until it abruptly drops to zero as the aerosol boundary moves past the point. The time required for the aerosol boundary to reach a point P located a distance h above the chamber floor is $(H - h)/V_{TS}$. The concentration becomes zero everywhere in the chamber after a time equal to H/V_{TS} has elapsed.

At the other extreme, stirred settling (Fig. 3.5b), the aerosol is stirred vigorously, and it is assumed that the concentration is always uniform throughout the chamber. Diffusion, resuspension, and deposition on the walls are assumed to be negligible. The particle settling velocity is superimposed on the vertical components of the convective velocity. Because the up and down components of the convective velocity are equal and cancel out over a time period that is long compared with their period, every particle will have an average *net* velocity equal to V_{TS}. The concentration decreases with time, but the rate of removal also decreases with time, because it is proportional to the number of particles left suspended in the chamber,

which, although the particles are uniformly mixed, is decreasing. The fraction removed, dn/n, during an interval of time dt that is brief compared with the time for convective motion is

$$\frac{dn}{n} = \frac{V_{TS}dt}{H}$$

Integrating for an initial condition of $n = n_0$ at $t = 0$ gives

$$\frac{n(t)}{n_0} = \exp\left(\frac{-V_{TS}t}{H}\right) \tag{3.36}$$

The concentration of particles undergoing stirred settling decays exponentially with time and, as such, never reaches zero. In stirred settling, the concentration reaches $1/e$ of the original concentration in the time required for complete removal in the tranquil settling situation (H/V_{TS}).

For polydisperse aerosols, the situation is more complicated. In both types of settling, the average size of the particles remaining in suspension decreases with time. For tranquil settling, there is no distinct upper boundary. For stirred settling, Eq. 3.36 can be applied to each particle size in the size distributions. Equations characterizing particle concentrations as a function of time can be obtained by integrating Eq. 3.36 over the size distribution.

EXAMPLE

The concentration of wood dust in a room with a 3-m ceiling height is 10 mg/m^3. What will be the concentration after one hour? Assume ideal monodisperse stirred settling of particles with aerodynamic diameters of 4 μm.

$$V_{TS} = \frac{\rho_0 d^2 g}{18\eta} = \frac{1000(4\times10^{-6})^2 9.81}{18(1.81\times10^{-5})} = 4.8\times10^{-4} \text{ m/s}$$

$$\left[V_{TS} = \frac{1(4\times10^{-4})^2 981}{18(1.81\times10^{-4})} = 4.8\times10^{-2} \text{ cm/s}\right]$$

$C_m \propto n$ for monodisperse aerosols, and $t = 60 \times 60 = 3600$ s. Hence,

$$C_{3600} = C_0 \exp\left(\frac{-V_{TS}t}{H}\right)$$

$$C_{3600} = 10 \exp\left(\frac{-4.8\times10^{-4}\times3600}{3}\right) = 10 \exp(-0.576) = 5.6 \text{ mg/m}^3$$

$$\left[C_{3600} = 10 \exp\left(\frac{-4.8\times10^{-2}\times3600}{300}\right) = 5.6 \text{ mg/m}^3\right]$$

3.9 INSTRUMENTS THAT RELY ON SETTLING VELOCITY

Several instruments rely on particle settling to measure the aerodynamic diameter of a particle or to fractionate particles into two or more size categories. The simplest is the *sedimentation cell*, which is similar to that adapted by Millikan for his oil-drop experiments. (See Section 15.3.) Aerosol particles are introduced into a sealed volume of less than 1 cm^3 and illuminated through windows by an intense beam of light. The particles are viewed by a horizontal microscope in a direction perpendicular to the light beam. The particles, which appear as tiny specks of light, are timed individually as they settle between calibrated lines marked on the eyepiece of the microscope. The true settling distance is known from calibration and is usually less than 1.0 mm. The aerodynamic diameter is calculated directly from the measured settling velocity.

The Brownian motion of particles less than 0.3 μm in diameter causes significant variation in the measured settling velocity, and such particles may wander out of the field of view. Particles larger than 5 μm settle too fast for accurate measurement. Care must be taken to minimize thermal convection caused by the heat of the light beam. Care must also be taken to prevent an operator bias toward the selection of larger and brighter particles.

For larger particles, 5–50 μm, the aerodynamic diameter can be measured directly using a settling tube (Wall et al, 1985). Typically, this is a vertical glass tube about 7 mm in diameter and 0.3–0.8 m long, illuminated along its axis by a low-power laser. The tube is marked off in 10-cm or other convenient intervals. Particles are seen as points of light, and their settling time over a known distance is measured with a stopwatch. The choice of tube diameter is a compromise between reducing convection currents inside the tube and reducing tube wall effects. This method provides an absolute measurement of the aerodynamic diameter and requires no calibration.

A *vertical elutriator* is a device used to remove particles larger than a certain aerodynamic diameter from an aerosol stream. The aerosol flows upwards at low velocity in a vertical duct. Particles having a V_{TS} greater than the duct velocity cannot be carried out of the duct and are thereby removed from the aerosol stream. The situation can be thought of as a contest between the particle and the airstream, with the faster one controlling the outcome ("winning"). These devices work satisfactorily for rough separation of large particles, but the distribution of gas velocities in the duct makes it difficult to obtain a precise cutoff. The large particles, having a net downward velocity, may "filter" out smaller particles moving upwards. A vertical elutriator used for cotton dust sampling is described in Section 11.5.

The *horizontal elutriator* can be used either as a separator to fractionate an aerosol stream or as an aerosol spectrometer to measure the distribution of particle size. For separation, an aerosol is passed at low velocity through a horizontal duct having a rectangular cross section. Particle settling is perpendicular to the gas streamlines. Particles reaching the floor of the duct are removed from the aerosol stream. This situation also may be viewed as a contest that results in an attenuation in particle concentration that is greater for the larger particles than for the smaller ones. As shown in Fig. 3.6, particles having a $V_{TS} > H\bar{V}_x/L$, where $\bar{V}_x = \bar{U}$, will be

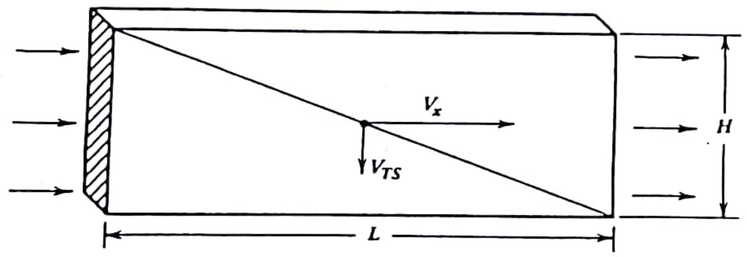

FIGURE 3.6 Horizontal elutriator, side view showing limiting trajectory for complete capture.

completely removed if the flow is laminar. Smaller particles will have a fraction $V_{TS}L/\overline{V}_xH$ of their number removed. Horizontal plates evenly spaced in the duct will reduce H and increase the number collected for a given particle size. The horizontal elutriator is one of the methods used for respirable mass sampling. (See Section 11.5.)

A similar approach is utilized in an elutriation spectrometer to measure the distribution of the particle sizes. Two flow streams are used: a clean airstream and an aerosol stream. A horizontal laminar flow of clean air at an average velocity $\overline{V}_x$, called the winnowing stream or carrier gas stream, is established in the duct (Fig. 3.7). At the beginning of the separation section, the aerosol is introduced as a thin stream or film along the upper surface of the duct. Particles of different sizes settle at different rates and deposit at different locations along the duct floor. There is a unique location along the floor where each particle size will deposit. The distance from the inlet to that location for a particle with a settling velocity V_{TS} is

$$L_d = \frac{H\overline{V}_x}{V_{TS}} \tag{3.37}$$

where H is the height of the duct. The bottom of the duct is lined with glass slides or foil that can be removed in sections and analyzed for the number or mass of particles deposited. Each section corresponds to a range of V_{TS} that defines a range

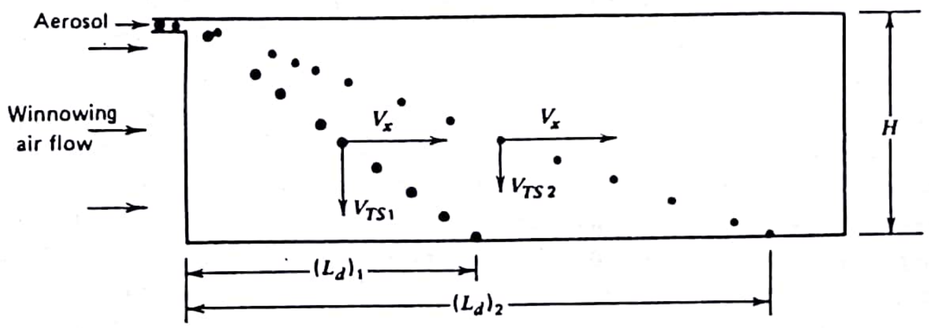

FIGURE 3.7 Horizontal elutriation aerosol spectrometer.

of aerodynamic diameters. The distribution of number (or mass) as a function of aerodynamic diameter is determined from the fraction of the total number (or mass) in each size range. A filter can be placed at the end of the duct to capture particles smaller than the limiting size. Because $V_{TS} \propto d_a^2$, the duct must be excessively long to collect small particles. One approach is to use a duct whose width increases along its length to reduce $\overline{V}_x$ and permit smaller particles to be separated in an instrument of convenient length. Because gravitational settling of aerosol particles is a slow process, the carrier gas flow rate must be low, and consequently, these instruments are sensitive to convection currents.

The principle of the *aerosol centrifuge* is similar to that of the horizontal elutriator except that the force of gravity is replaced by centrifugal force. The elutriation duct is wrapped around a cylinder with the collection surface at the periphery and is rotated. The duct is often arranged in a spiral to permit the aerosol and carrier gas air to be introduced along the rotational axis and exit at the periphery. The outer side of the channel is lined with a deposition foil, which is removed, segmented, and analyzed for particle mass to determine the mass distribution as a function of aerodynamic diameter. Because centrifugal force can easily be made much greater than the force of gravity, these devices can operate at a higher sampling flow rate and separate smaller particle sizes than the horizontal elutriator can. They are practical for particle sizes of 0.1–15 μm aerodynamic diameter. The radial component of flow velocity causes a secondary flow (eddies), which distorts the flow stream and limits the resolution for small particle sizes. Some aerosol centrifuges are high-resolution instruments that can separate particles differing in aerodynamic diameter by only a few percent. Characteristics of aerosol centrifuges are summarized by Marple et al. (1993).

3.10 APPENDIX: DERIVATION OF STOKES'S LAW

The Navier–Stokes equation for incompressible flow (flow for which $\nabla \cdot \mathbf{V} = 0$) can be written, neglecting gravity and buoyancy forces, as

$$\rho\left(\frac{\partial U}{\partial t} + u\frac{\partial U}{\partial x} + v\frac{\partial U}{\partial y} + w\frac{\partial W}{\partial z}\right) = -\frac{\partial p}{\partial x} + \eta\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}\right) \tag{3.38}$$

for the x direction, and similar equations can be written for the y and z directions. In operator notation, this becomes

$$\rho\frac{D\mathbf{V}}{dt} = -\nabla p + \eta\nabla^2(\mathbf{V}) \tag{3.39}$$

where $\mathbf{V}$ is the three-dimensional velocity vector of the fluid at a fixed point (x, y, z) or (r, θ, ϕ) in space and $D\mathbf{V}/dt$ *is* the substantial (total) derivative of $\mathbf{V}$ (i.e., $D\mathbf{V}/dt = \partial\mathbf{V}/\partial t + \mathbf{V} \cdot \nabla\mathbf{V}$). Equation (3.39) is called the *Eulerian description of fluid motion*. For steady flow, the first term in the substantial derivative is zero. To use

the Navier–Stokes equation in this derivation, we assume that the particle is held fixed and the fluid is moved past it at a velocity $V = V_\infty$, where V_∞ is the velocity of the fluid well away from the particle. The fluid velocity is assumed to be in the z direction in spherical coordinates. (See Fig. 3.8.) The $\mathbf{V} \cdot \nabla \mathbf{V}$ term is the acceleration (inertial) term and represents the resistance to flow caused by the inertia of the gas that has to be accelerated around the particles. Because Stokes's derivation is restricted to low Reynolds numbers, the inertial terms are small compared with the viscous terms and can be neglected. Thus, we get the simplified (linear) form of the Navier–Stokes equation for steady, incompressible flow at low Reynolds number:

$$\nabla p = \eta \nabla^2 \mathbf{V} \tag{3.40}$$

The analytical solution of this equation, giving the flow field around a sphere of radius a in spherical coordinates (see Fig. 3.8), has been found. [See, for example, Bird et al. (1960) or Landau and Lifshitz (1959).] The velocity distribution is

$$V_r = V_\infty \left[1 - \frac{3}{2}\left(\frac{a}{r}\right) + \frac{1}{2}\left(\frac{a}{r}\right)^3 \right] \cos\theta \tag{3.41}$$

$$V_\theta = -V_\infty \left[1 - \frac{3}{4}\left(\frac{a}{r}\right) - \frac{1}{4}\left(\frac{a}{4}\right)^3 \right] \sin\theta \tag{3.42}$$

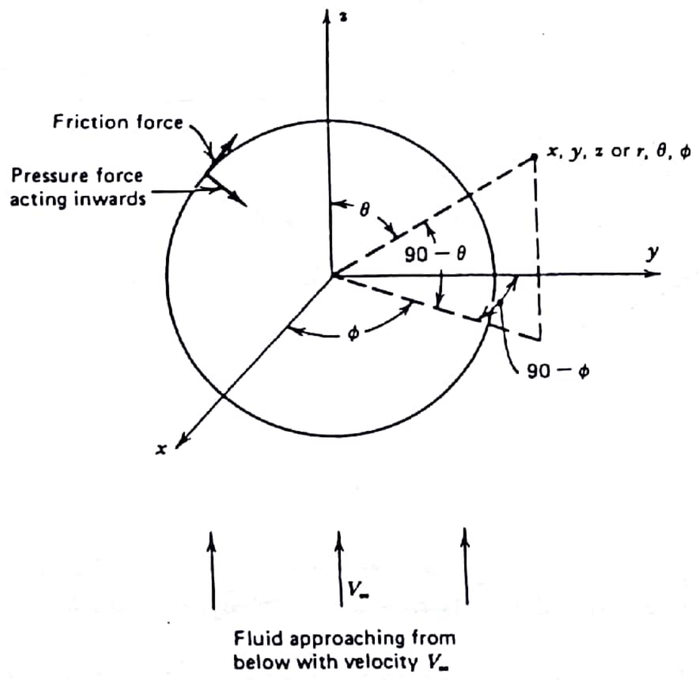

FIGURE 3.8 Forces acting on a stationary sphere in a flowing fluid.

Because of symmetry, V is the same for any value of ϕ; that is, V does not depend on ϕ. The velocity distribution satisfies the boundary conditions, namely, $V_r = V_\theta = 0$ at the surface of the sphere ($r = a$), and V_r and V_θ go to V_∞ for $r >> a$. This velocity distribution, Eqs. 3.41 and 3.42, causes normal and tangential forces p and τ, respectively. At any point in the fluid near the sphere, these forces are given by

$$p = p_0 - \frac{3\eta V_\infty}{2a}\left(\frac{a}{r}\right)^2 \cos\theta \tag{3.43}$$

$$\tau = \frac{3\eta V_\infty}{2a}\left(\frac{a}{r}\right)^4 \sin\theta \tag{3.44}$$

where p_0 is the ambient pressure. The normal force p goes to p_0 for $r >> a$, as expected.

To calculate the net force exerted by the fluid on the sphere, we integrate the pressure (normal) force and friction (tangential) force over the surface of the sphere ($r = a$).

The z component of the pressure force is $-p \cos\theta$, which is multiplied by an incremental area (see Fig. 3.9) on the surface of the sphere, $(a \sin\theta\, d\phi)(a\, d\theta) = a^2 \sin\theta\, d\theta\, d\phi$, and integrated over the surface to give the total pressure force. Substituting $r = a$ into Eq. 3.43 and setting up the integration gives

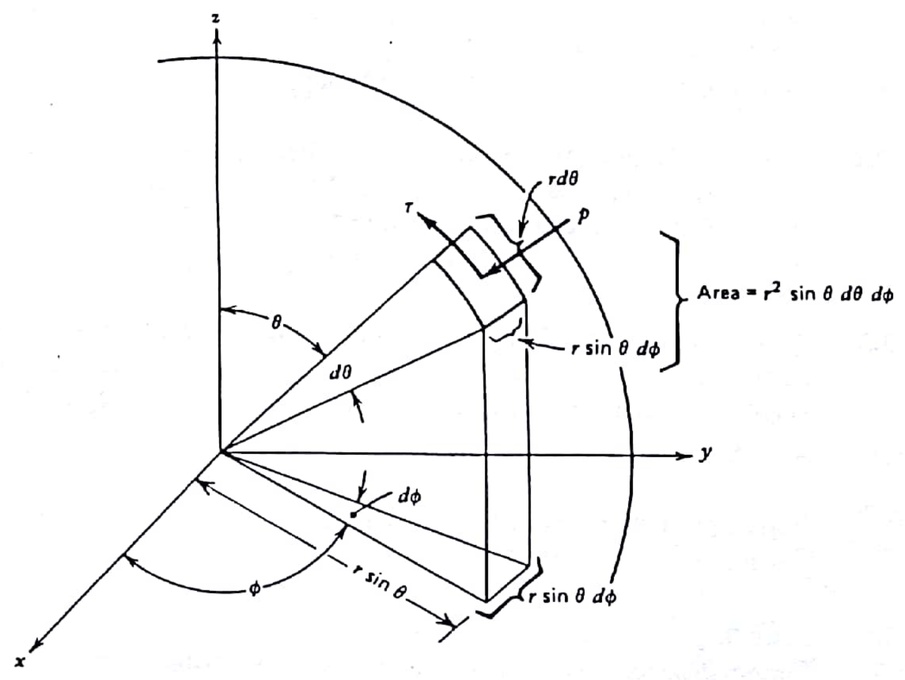

FIGURE 3.9 Diagram for the integration of forces.

$$F_n = \int_S -p \cos\theta \, dS = \int_0^{2\pi}\int_0^{\pi}\left(-p_0 + \frac{3\eta V_\infty}{2a}\cos^2\theta\right)a^2 \sin\theta \, d\theta \, d\phi \tag{3.45}$$

Completing the integration gives the form component of the Stokes drag:

$$F_n = 2\pi\eta V_\infty a \tag{3.46}$$

The tangential shear stress τ acts tangentially at the surface of the sphere. The component in the z direction is $\tau \sin\theta$. Integration of this force over the surface of the sphere gives the net frictional component of the Stokes drag,

$$F_\tau = \int_S \tau \sin\theta \, ds = \int_0^{2\pi}\int_0^{\pi}\left(\frac{3}{2}\frac{\eta V_\infty}{a}\sin^3\theta\right)a^2 \, d\theta \, d\phi \tag{3.47}$$

$$F_\tau = 4\pi\eta V_\infty a \tag{3.48}$$

The form and friction components are combined to give the total force acting on the particle due to its motion relative to the fluid, or Stokes's law:

$$F_D = 3\pi\eta V_\infty d \tag{3.49}$$

PROBLEMS

3.1 What is the drag force on a 50-μm-diameter particle traveling at 1 m/s in still air? Use the graph of C_D versus Re, Fig. 3.1. What is the drag force when the particle is traveling at 0.01 m/s?
ANSWER: 1.2×10^{-8} N, 8.2×10^{-11} N [1.2×10^{-3} dyn, 8.2×10^{-6} dyn].

3.2 Calculate the drag force on a 12-μm-diameter particle moving at 0.1 m/s using Newton's drag equation and Fig. 3.1. Check your result by calculating the drag force using Stokes's equation.
ANSWER: 2.0×10^{-10} N [2.0×10^{-5} dyn].

3.3 An old occupational hygiene rule of thumb is that a 10-μm silica (quartz) particle settles at a rate of 1 ft/min (0.5 cm/s). What is the true settling velocity for such a particle? The density of silica is 2600 kg/m³ [2.6 g/cm³].
ANSWER: 0.58 cm/s.

3.4 Derive an expression for the aerodynamic diameter of a cylindrically shaped particle of diameter D and length L in terms of D, L, χ, ρ_p.

3.5 For a 0.15-μm-diameter spherical particle (ρ_p = 2500 kg/m³), determine the Cunningham correction factor and the terminal settling velocity at standard conditions.
ANSWER: 2.11, 3.6×10^{-6} m/s [3.6×10^{-4} cm/s].

3.6 Calculate the terminal settling velocity of a 0.2-μm steel sphere (ρ_p = 7600 kg/m³) at atmospheric pressure and a temperature of (a) 20°C and (b) 200°C.
ANSWER: 1.7×10^{-5} m/s, 1.5×10^{-5} m/s [1.7×10^{-3} cm/s, 1.5×10^{-3} cm/s].

3.7 In 1883 the volcano Krakatoa exploded, injecting dust 32 km up into the atmosphere. Fallout from this explosion continued for 15 months. If one assumes that the settling velocity of the particles was constant and neglects slip correction, what was the minimum particle size present? Assume that the particles are rock spheres with a density of 2700 kg/m³.
ANSWER: 3.2 μm.

3.8 What is the settling velocity of a 0.5-μm silica sphere at 20°C and an ambient pressure of 150 kPa? The density of silica is 2600 kg/m³.
ANSWER: 2.4×10^{-5} m/s [0.0024 cm/s].

3.9 Air is dried by bubbling it through concentrated sulfuric acid (ρ_p = 1.84 g/cm³). A l-m tube 0.1 m in diameter contains 1.5 L of acid. When the bubbles burst at the liquid surface, they form droplets. What is the largest droplet that can be carried out of this system? The airflow rate is 10 L/min. Neglect slip correction.
ANSWER: 19.5 μm.

3.10 Calculate the equivalent volume diameter, the aerodynamic diameter, the terminal settling velocity, and the drag force at terminal velocity for (a) a 2-μm-diameter standard-density sphere and (b) a doublet (cluster of two) composed of 2-μm-diameter standard-density spheres. Neglect slip correction. Compare your results in a table.
ANSWER:

	d_e (μm)	d_a (μm)	V_{TS} (m/s [cm/s])	F_D (N [dyn])
(a)	2	2	1.2×10^{-4} [0.012]	4.1×10^{-14} [4.1×10^{-9}]
(b)	2.52	2.38	1.7×10^{-4} [0.017]	8.2×10^{-14} [8.2×10^{-9}]

3.11 Derive the expression

$$d_a = n^{1/3}\left(\frac{\rho}{\chi}\right)^{1/2} d$$

where d_a is the aerodynamic diameter of a *cluster* of n spheres, each having a diameter d and density ρ, and χ is the shape factor for the cluster.

3.12 What is the drag force on a doublet composed of 2-μm-diameter spheres traveling at its terminal settling velocity? The particle density is 3000 kg/m³.

Neglect slip correction. (*Hint*: There is an easy way to do this; review the derivation of Eq. 3.13.)
ANSWER: 2.5×10^{-13} N [2.5×10^{-8} dyn].

3.13 What is the settling velocity of an asbestos fiber in the shape of a cylinder 1 μm in diameter and 10 μm long at standard conditions? Assume random orientation and a density of 2500 kg/m³, and neglect slip correction.
ANSWER: 3.2×10^{-4} m/s [0.032 cm/s].

3.14 Air with a uniform velocity U is flowing through a settling chamber with dimensions L, H, and W, as shown in Fig. 3.10. Assume that there are no internal partitions, there is no reentrainment, the flow is laminar, the aerosol is uniformly distributed in the incoming air, and slip correction is negligible. Derive an expression that gives the diameter of the smallest particle that will be collected with 100% efficiency. This diameter will be a function of U, L, H, η, g, and ρ_p.

3.15 (a) Derive an expression for the collection efficiency (as a function of particle size) for a multilayer settling chamber (Fig. 3.10) with N equispaced sections. Assume uniform flow, a uniform distribution of aerosol particles at the inlet, and negligible slip correction. (b) Show that the collection efficiency for a given particle size is proportional to the total area of the collection surfaces.

3.16 Air is flowing with a uniform laminar flow in a horizontal duct with a square cross section. The duct is 1 ft × 1 ft × 100 ft long, and the air flows at a rate of 1000 ft³/min. If an aerosol of 10-μm iron-oxide spheres (ρ_p = 5.2 g/cm³) is uniformly distributed across the inlet, what fraction of entering particles will leave the 100-foot section? Assume that a particle sticks if it hits the bottom of the duct.
ANSWER: 0.69.

3.17 It is desired to recover particles larger than 200 μm emitted by a cement kiln, using a horizontal-flow settling chamber having a residence time of 24 s.

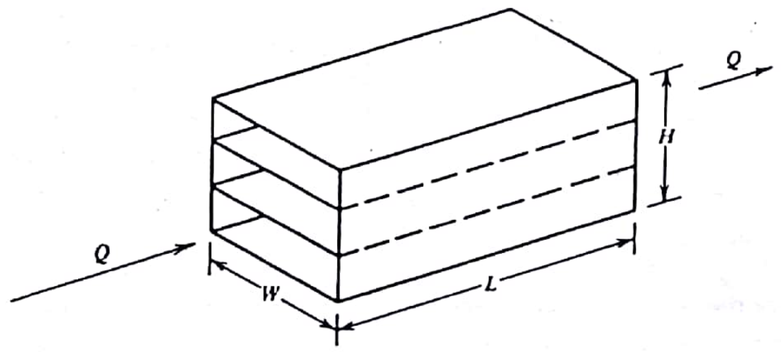

FIGURE 3.10 Diagram of settling chamber.

What is the maximum height of the chamber? Assume ρ_p = 3100 kg/m³ [3.1 g/cm³].
ANSWER: 39 m.

3.18 What is the terminal settling velocity of a 100-μm-diameter lead sphere (ρ_p = 11,300 kg/m³) in still air?
ANSWER: 1.8 m/s [180 cm/s].

3.19 What particle size (ρ_p = 4000 kg/m³) is too large to be carried out of a vertical exhaust duct 14 inches in diameter that handles 200 ft³/min?
ANSWER: 111 μm.

3.20 What is the terminal settling velocity of a 40-μm-diameter uranium sphere (ρ_p = 19,000 kg/m³)?
ANSWER: 0.74 m/s [74 cm/s].

3.21 A spherical steel particle (ρ_p = 7800 kg/m³) is observed to settle in still air at a rate of 2.0 m/s. What is its diameter?
ANSWER: 134 μm.

3.22 You measure airborne particulate concentrations of 20 and 6 mg/m³ one and two hours, respectively, after an explosion in a well-stirred reactor vessel 14 ft high. Assuming simple stirred settling of a monodisperse aerosol, what was the concentration immediately after the explosion? What is the particle size (aerodynamic diameter)?
ANSWER: 65 mg/m³, 6.9 μm.

3.23 What change in concentration occurs during a 5-min interval for 10-μm particles undergoing stirred settling in a chamber 1.0 m high? Assume that the particle density is 3000 kg/m³ [3 g/cm³].
ANSWER: 93%.

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